Deconstructing Black-Litterman: How to Get the Portfolio You Already Knew You Wanted[1]

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[1] We wish to acknowledge helpful comments by Paul Erlich.

ABSTRACT

The Markowitz (1952, 1959) mean-variance (MV) efficient frontier has been the theoretical standard for defining portfolio optimality for more than a half century. However, MV optimized portfolios are highly susceptible to estimation error and difficult to manage in practice (Jobson and Korkie 1980, 1981; Michaud 1989). The Black and Litterman (BL) (1992) proposal to solve MV optimization limitations produces a single maximum Sharpe ratio (MSR) optimal portfolio on the unconstrained MV efficient frontier based on an assumed MSR optimal benchmark portfolio and active views. The BL portfolio is often uninvestable in applications due to large leveraged or short allocations. BL use an input tuning process for computing acceptable sign constrained solutions. We compare constrained BL to MV and Michaud (1998) optimization for a simple data set. We show that constrained BL is identical to Markowitz and that Michaud portfolios are better diversified under identical inputs and optimality criteria. The attractiveness of the BL procedure is due to convenience rather than effective asset management and not recommendable relative to alternatives.

EXECUTIVE SUMMARY

Since the publication of their original article in 1992, Black-Litterman (BL) has become a popular method in practical finance for creating superficially stable portfolios, adjusted to investor views. A popular perception is that BL can solve the instability problems of portfolios on Markowitz efficient frontiers. In fact, the instability issues of Markowitz portfolios are caused by estimation error (Michaud 1998, 2008), which BL does nothing to explicitly handle. The BL method assumes a perfectly known market portfolio in a state of undisturbed equilibrium, a perfectly known covariance matrix, and correct investor views numerically calibrated to perfectly quantify the exogenous knowledge of the investor. On top of these heroic assumptions, the BL formula itself is built on faulty statistical theory and is not optimal in any mathematical sense. Besides, since it is equivalent to a maximum Sharpe ratio Markowitz optimization with specific inputs, it inherits all of the instability of Markowitz optimization, especially when the frontier is extended beyond the BL portfolio.

Black and Litterman (1992) give a tuning parameter τ to adjust the strength of the views. This parameter may be fixed or adjusted, and is in practice often used to guarantee investable portfolios. Adjusting τ for investability amounts to either adjusting the data to fit the desired solution or adjusting one's "exogenous" views, and is a violation of fundamental principles of statistical analysis. Like the unadjusted BL portfolio, the τ-adjusted portfolio can also be found on a Markowitz frontier with particular inputs and inherits the properties and shortcomings of that method.

In our article, we provide a simple but detailed example of a realistic Black-Litterman analysis and show the corresponding Markowitz inputs and frontiers which contain the BL portfolios. Moving away from the BL portfolios at their maximum Sharpe ratio points, these frontiers veer quickly into uninvestable portfolios with short and/or leveraged positions in some assets and are not useful to managers who require access to multiple risk profiles tailored to investors' risk preferences. The BL portfolios and frontiers in our example are compared with better solutions created with methods that explicitly account for estimation error. Michaud efficient portfolios are better diversified and more intuitive, have superior out-of-sample performance by design, and do not rely on false assumptions or dial in a preordained result.

Users of Black-Litterman or its implied returns should be mindful of these methods' limitations. BL does not solve but rather conceals the instability and estimation error problems of Markowitz mean-variance optimization. Because it is not a proper optimization method and tends to assign too much confidence to personal views it may often miss useful information while exposing investors to unnecessary risk. The simplicity and apparent adequacy of the procedure comes at the peril of ignoring better statistically-based methods that merge all of the available information into a more effective portfolio creation process.

INTRODUCTION

For more than a half century, the Markowitz (1952, 1959) mean-variance (MV) efficient frontier has been the theoretical standard for defining linear constrained portfolio optimality. Markowitz optimization is a convenient framework for computing MV optimal portfolios that are designed to meet practical investment mandates.

MV optimization, however, has a number of well-known investment limitations in practice. Optimized portfolios are unstable and ambiguous and highly sensitive to estimation error in risk-return estimates. The procedure tends to overweight/underweight assets with estimate errors in high/low means, low/high variances, and small/large correlations, often resulting in poor performance out-of-sample (Jobson and Korkie 1980, 1981). MV optimized portfolios in practice are often investment unintuitive and inconsistent with marketing mandates and management priors. Ad hoc input revisions and constraints result in an MV optimization process that is largely an exercise in finding "acceptable" rather than optimal portfolios (Michaud 1989).

To address estimation error issues in MV optimization, Black and Litterman (BL) (1992) propose a single Maximum Sharpe Ratio (MSR) portfolio. This portfolio is constructed assuming: an unconstrained MV optimization framework; an assumed MSR optimal benchmark or "market" portfolio; an estimation error free covariance matrix; and active investor views. BL optimal portfolios often have large leveraged and/or short allocations that may make them uninvestable in applications. Moreover, Jobson and Korkie document severe out-of-sample investment limitations for the unconstrained MV optimization framework, of which BL optimization is an example.[2]

BL introduce an input "tuning" parameter τ that enables sign constrained MSR optimal solutions.[3] We describe the mathematical properties of BL, including τ-adjustment, and use a simple dataset to illustrate the procedures. We show that the BL sign constrained portfolio is identical to Markowitz MSR for the same inputs and consequently no less estimation error sensitive. BL optimality is also benchmark centric and subject to the Roll (1992) critique of optimization on the wrong efficient frontier. The Michaud (1998) proposal to address estimation error uses Monte Carlo resampling and frontier averaging methods to generalize the Markowitz efficient frontier.[4] We compare BL and Michaud

[2] See also deMiguel et al (2009) for an empirical study on the same issue.

[3] Extensions of the BL formula include different types of nonlinear views, alternative view specifications, asset classes, and return distributions (e. g. Meucci 2008a,b).

[4] Michaud optimization was invented and patented by Richard Michaud and Robert Michaud, U.S. patent 6,003,018. (Michaud 1998, Michaud and Michaud 2008a,b), worldwide patents pending. New Frontier Advisors is exclusive worldwide licensee.

MSR optimized portfolios under identical assumptions and show that the Michaud portfolios are better diversified and risk managed.[5]

Section I describes the mathematical characteristics and statistical issues associated with the Black-Litterman procedure including τ-adjustment. Section II illustrates BL optimization with a simple data set, compares the portfolios to Markowitz and Michaud alternatives, and demonstrates the sensitivity to covariance estimation error. Section III discusses BL relative to benchmark centric optimization, unconstrained MV framework, and investor risk aversion. Section IV summarizes and concludes.

I. BLACK AND LITTERMAN OPTIMIZATION

I.A. Black-Litterman Framework

Black-Litterman optimization requires three investment assumptions: 1) unconstrained MV optimization; 2) capital market portfolio $\boldsymbol{M}$ in "equilibrium" on the Markowitz MV efficient frontier; 3) covariance matrix $\boldsymbol{\Sigma}$ without estimation error. Under these conditions $\boldsymbol{M}$ is the MSR portfolio on the MV efficient frontier. Unconstrained MV optimization and perfectly estimated covariance matrix allow computation of the "implied" or "inverse" returns $\boldsymbol{\Pi} = \boldsymbol{\Sigma}\boldsymbol{M}$ consistent with MV Sharpe ratio optimality (Sharpe 1974, Fisher 1975). The result is a set of estimated returns $\boldsymbol{\Pi}$ and covariance matrix $\boldsymbol{\Sigma}$ for which the market portfolio $\boldsymbol{M}$ is the MSR efficient portfolio on the unconstrained MV efficient frontier.[6]

In elemental form, the BL proposal is a rationale for the identification of a benchmark portfolio to anchor the optimization and overlay investment views. Benchmark anchoring of MV optimized portfolios has a long tradition in investment practice and is subject to Roll (1992) critiques.[7] The procedure trivially replicates its input absent any additional investor views. Investor views are processed using an adaptation of the Theil and Goldberger (1961) mixed estimation formula relative to the implied return estimates $\boldsymbol{\Pi}$.[8]

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[5] Michaud and BL have been discussed as competing procedures (e.g. deFusco 2001).

[6] The use of implied or inverse returns as "default" return estimates in portfolio optimization is not recommendable. Inverse returns are a function of the covariance matrix which, by definition, is devoid of return information. Inverse returns function solely to reverse-engineer the unconstrained MV optimization and negate any optimality properties bestowed by that optimization. Inverse returns are not unique and not on the same scale as actual forecast returns; a positive scalar multiple is also an inverse return. Inverse returns require MSR optimality of the market portfolio, which is unknown and highly unlikely a priori. Inverse returns require an unconstrained MV optimization framework, which is unrealistic for practical investment. In Section III we further investigate the effects of estimation error in the covariance matrix on the BL process.

[7] See e.g. Michaud and Michaud (2008, Ch. 9) and references.

[8] The Theil-Goldberger formula is presented as a way to combine information extracted from data with exogenous information in a regression. Mean estimation can be viewed as an intercept-only regression. The derivation of the formula relies on the sampling distribution of the mean estimate as a normal distribution centered around the sample mean and sample variance scaled down according to the number of observations. Since implied returns do not have this distribution, the derivation of the formula for adjusting the estimates to

The revised returns with views $\boldsymbol{\mu}_{BL}$ are used to compute the BL MV optimal portfolio *B*. Deviations from the index weights indicate optimal overweights and underweights for each asset relative to the benchmark portfolio.

I.B. Black-Litterman Mathematical Structure

It is useful to briefly review the mathematical structure of the BL optimization framework.[9] We are given data for *N* assets with theoretical mean $\boldsymbol{\mu}$ and known variance $\boldsymbol{\Sigma}$. We assume $\boldsymbol{M}$ a vector of "equilibrium" market or index portfolios weights. We construct an estimate of the "implied" or "inverse" expected returns $\boldsymbol{\Pi} = \boldsymbol{\Sigma M}$. $\boldsymbol{\Pi}$ represents the returns associated with market portfolio $\boldsymbol{M}$ in equilibrium for known covariance matrix $\boldsymbol{\Sigma}$.

Views are specified as $\boldsymbol{P\mu} \sim \mathrm{N}(\boldsymbol{v}, \boldsymbol{\Omega})$, where $\boldsymbol{P}$ is a *K* x *N* matrix whose rows are portfolios with views, $\boldsymbol{v}$ is the vector of expected returns for these portfolios, and $\boldsymbol{\Omega}$ is the covariance matrix for the views. In the terminology of Bayesian statistics, we assign the views as the prior distribution. BL introduce a tuning parameter τ to adjust the impact of the views. They express the distribution of the equilibrium mean as $\mathrm{N}(\boldsymbol{\Pi}, \boldsymbol{\tau\Sigma})$. The parameter τ may be viewed as a proxy for $1/T$, the reciprocal of the number of time periods in the data, or as a measure of the relative importance of the views to the equilibrium but is often used simply to find investable (long-only) BL optimal portfolios. The resulting posterior distribution then has a normal distribution with mean equal to the BL estimates which can be expressed as:

$$\boldsymbol{\mu}_{BL} = \boldsymbol{\Pi} + \boldsymbol{\Sigma}\mathbf{P}' \left(\frac{\boldsymbol{\Omega}}{\tau} + \mathbf{P\Sigma P'}\right)^{-1} (\boldsymbol{v} - \boldsymbol{P\Pi}) = \boldsymbol{\Pi} + \mathbf{V}. \tag{1}$$

The formulation (1) is useful, since it decomposes the estimate into the original data-based estimate $\boldsymbol{\Pi}$ and the contribution from the views $\mathbf{V}$. In fact, if the mean estimate is the vector of equilibrium implied returns $\boldsymbol{\Pi}$, the maximum information ratio unconstrained portfolio optimization results in portfolio weights $\boldsymbol{P_{BL}}$***,** which are proportional to:

$$\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{\mathrm{BL}} = \boldsymbol{M} + \mathbf{P}' \left(\frac{\boldsymbol{\Omega}}{\tau} + \mathbf{P\Sigma P'}\right)^{-1} (\boldsymbol{v} - \boldsymbol{P\Pi}) \tag{2}$$

The second term in the right hand side of equation (2) is a multiplication of the matrix $\mathbf{P}'$, whose *k* columns are the portfolios with views, by the *k* by 1 vector $\left(\frac{\boldsymbol{\Omega}}{\tau} + \mathbf{P\Sigma P'}\right)^{-1} (\boldsymbol{v} -$

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the exogenous views is invalid for the BL case. Plugging the implied returns into the Theil-Goldberger formula is an approximation with unknown bias and error properties.

[9] We use the traditional notation of the BL literature (e. g. Black and Litterman 1992, Meucci 2008a).

$\boldsymbol{P\Pi}$). Thus, in mathematical terms, the contribution of investor views to the BL portfolio is confined to the subspace spanned by the view portfolios. More intuitively, the BL portfolio pushes itself towards or away from each view as necessary, but is limited to directions specified by the view portfolios themselves. Active bets induced by including views result in allocations in a direction which is solely a linear combination of those views.

I.C. Investable Black-Litterman Portfolios

BL unconstrained MV optimized portfolios often possess large leveraged and/or short positions. In practice, investors often require that optimal portfolios are investable; i.e., that they are sign constrained and/or linear inequality constrained within some specified range.[10] By definition the equilibrium or market portfolio is sign constrained. BL introduce the input "tuning" parameter $0 \leq \tau \leq 1$ for finding portfolios between the BL portfolio $\boldsymbol{B}$ and index portfolio $\boldsymbol{M}$ that are non-negative (long-only) or satisfy some suitable inequalities. The parameter τ provides a mechanism for finding investable BL portfolios. The parameter operates as a scalar that divides the variances associated with the uncertainty of the views. Smaller values of τ cause greater inflation of the views' uncertainty and limit their influence on the results. As τ is reduced, or the constant multiplier of the standard deviations of the views increased, the BL portfolio approaches the benchmark portfolio. The value of τ may be chosen to compute an investable BL portfolio when it uses just enough of the certainty in investor's views to meet investability constraint boundaries. The net effect of the τ-adjustment is to reduce the impact of the covariance matrix on the BL portfolio. At sufficiently high certainty, the procedure essentially ignores the covariance matrix and the optimization framework.

Without recourse to a formal τ-adjustment, an investor may define an investable BL portfolio simply by sufficiently increasing the standard deviations of any or all of the views. However, such a process is clearly ad hoc. Indeed, in a caricature of the BL procedure, some software providers have a "dial-an-optimal" option for each view so that an investor can create whatever BL portfolio desired. In this case, BL is simply optimization by definition with little regard to investment value.

I.D. Further Comments on the Black-Litterman Statistical Framework

We note that no standard statistical procedure produces the implied returns as estimates for portfolio expectations. The formula comes out of theoretical assumptions of market efficiency and equilibrium, and the additional assumptions of current efficient equilibrium market weights and an error-free covariance matrix. Since estimation of the mean is

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[10] See Markowitz (2005) for a discussion of why inequality constraints are fundamentally important for financial theory as well as practical application.

ancillary to estimation of variance, using a mean estimate that is a function of the covariance estimate is tantamount to believing that there is no information for the first moment of the data, while there is perfect information for the second moment, the expected squared deviations from the unknowable first moment. From a pure data analysis perspective this assumption is untenable. The usual meanings of mean and variance have been lost, and the significance of these calculations is unclear other than as convenient inputs to an optimizer that has been designed to produce preordained answers.

Additionally, the τ adjustment itself is an ad hoc modification of the prior distribution to steer the outcome towards some desirable result, which violates the principles of a rigorous Bayesian analysis. The problem is that one does not change one's internal beliefs to modify an outcome when confronted with those beliefs. The only alternative characterization of τ adjustment is as a modification of the scale of the covariance of the data, which is also changing the model during the model-fitting stage. The adjustment of τ to attain investability is an intervention which contaminates the rigor of the analysis and must be viewed as an ad hoc correction of a flawed procedure, and a major departure from rigorous statistical analysis.

II. BLACK-LITTERMAN OPTIMIZATION ILLUSTRATED

II.A. Risk-Return Inputs and Investor Views

Institutional asset allocation often includes twenty or more asset classes. For pedagogical clarity and simplicity, we use the eight asset class dataset described in Michaud (1998) to illustrate the characteristics of BL, Markowitz, and Michaud optimized portfolios.[11] The eight asset classes in the dataset are displayed in column one of Table 1 and consist of two bond and six country equity indices. The historical annualized risk-return estimates are based on eighteen years of monthly returns and given in columns three and four of Table 1. The correlations are given in Table 1A in the appendix.

The Michaud dataset reflects a simple global index universe. We define a market portfolio as a 60/40 asset mix of domestic and international stocks and bonds. For simplicity we equal weight the bond indices, equal weight U.S. vs. non-U.S. equity indices and equal weight the non-U.S. indices. Our market portfolio allocations are given in column two in Table 1.[12] The BL implied or inverse mean returns from the Sharpe-Fisher

[11] This dataset is widely available and has been used in a number of estimation error optimization studies. We note the pedagogical simplicity of a relatively small generic set of assets may minimize the instability that often exists in institutional asset allocation portfolios, particularly with respect to estimation error in the covariance matrix which accumulates quickly as the number of assets increases.

[12] We note that this "market" portfolio is in fact very close to in-sample MV efficient for the data.

procedure for the assumed market portfolio and covariance matrix are given in column five in Table 1.

BL optimization requires at least one active investor view to avoid being self-referential. The Theil-Goldberger procedure used in BL is illustrated with the data in column eight in Table 1. Assume a U.S. investor skeptical of European equities. In our illustration we posit a 5% return premium for the U.S. versus European equity indices with a 5% uncertainty level (standard deviation). In our example the investor feels that France bears slightly more of the brunt of this negative view on European equities so France is assigned 40% weight while Germany and the U. K. are each assigned 30%. The statistics at the bottom of column eight in Table 1 show the following information: the BL mean and historical covariance implies a US/EU equity premium of -1.5% with standard deviation of 16.7%. The view updates the return estimates resulting in a 4.5% return premium and a 6.0% increase. The BL+view mean, given in column six Table 1, increases US return from 8.5% to 10%, while the returns for the three European indices are reduced from 10.9%, 8.6%, and 10.0% to 5.5%, 3.8%, and 7.3%.

Table 1
Risk-Return Inputs and Investor Views

Asset Name	Market Portfolio	Mean	Standard Deviation	BL Mean	BL+View Mean		Investor Views
Euro Bonds	20.0%	3.2%	5.4%	2.2%	2.2%		0.0%
US Bonds	20.0%	3.0%	7.0%	2.6%	2.6%		0.0%
Canada	6.0%	4.6%	19.0%	9.2%	9.6%		0.0%
France	6.0%	10.5%	24.4%	10.9%	5.5%		-40.0%
Germany	6.0%	6.4%	21.5%	8.6%	3.8%		-30.0%
Japan	6.0%	10.5%	24.4%	7.8%	4.9%		0.0%
UK	6.0%	9.5%	20.8%	10.0%	7.3%		-30.0%
US	30.0%	8.5%	14.9%	8.5%	10.0%		100.0%
						View Prior Return	5.0%
						View Prior Std Dev	5.0%
						Data Return	-1.5%
						Data Std Dev	16.7%
						View Posterior Return	4.5%
						View Effect on Return	6.0%

II.B. Optimized Portfolios

The standard risk-return estimates associated with the BL optimization procedure are given by the means in column six and standard deviations in column four in Table 1 with correlations given in Table 1A in the appendix. The market portfolio is reproduced from Table 1 in column two in Table 2. The BL+view means are reproduced in column three in Table 2. The resulting BL optimal portfolio including investor views is given in column four in Table 2.

BL optimization is, by definition, market portfolio centric. As noted earlier, only asset allocations in the view in Table 2 are affected by BL optimization relative to market weights. Market portfolio anchoring is the source of a perception of stability in the BL optimization process.

Table 2
BL Means and Optimized Portfolios

Asset Name	Market	BL+View Means	BL Optimal	BL* Means	BL* Optimal
Euro Bonds	20.0%	2.2%	20.0%	2.2%	20.0%
US Bonds	20.0%	2.6%	20.0%	2.6%	20.0%
Canada	6.0%	9.6%	6.0%	9.4%	6.0%
France	6.0%	5.5%	-6.5%	8.3%	0.0%
Germany	6.0%	3.8%	-3.4%	6.3%	1.5%
Japan	6.0%	4.9%	6.0%	6.4%	6.0%
UK	6.0%	7.3%	-3.4%	8.7%	1.5%
US	30.0%	10.0%	61.2%	9.2%	45.0%
Return	6.1%		7.2%		5.4%
Risk	9.6%		10.3%		9.5%

As in this example, the BL optimal portfolio often includes large positive and short asset allocations and is often uninvestable in applications. In many cases, investability requires a non-negative (long-only) optimized portfolio. BL recommend using the input tuning parameter τ to compute a sign constrained portfolio BL*. The τ procedure involves increasing the uncertainty or the standard deviation of the view. Reducing confidence in the view results in a portfolio part way between the (presumably non-negative) market portfolio and the original BL optimal. With just enough uncertainty the new optimal portfolio, BL*, will be non-negative (long-only) as well.

The τ-adjusted view means and optimized portfolio BL* that minimally reduce the confidence level of the inputs to find a "sign constrained BL optimal portfolio" are given in columns five and six in Table 2.[13] In this case the BL* solution reduces the allocations to European equities uniformly away from BL optimal weights and adds the (negative) sum to the U.S. allocation. The sign constrained BL* optimized portfolio is often the recommended BL optimal portfolio in applications.[14]

II.C. Black-Litterman and Markowitz Optimization

For sign constrained optimality, the relationship between BL and Markowitz is direct.[15] Given the BL* inputs the Black-Litterman MSR portfolio is exactly the same as the Markowitz MSR portfolio. Consequently, BL is no less sensitive to estimation error than MV optimization.[16]

We note that, in practice, the software for computing correctly τ-adjusted BL* means may be unavailable. As a consequence some analysts may be tempted to use the MSR sign constrained Markowitz portfolio with BL+view means as inputs. This will differ from the τ-adjusted "sign constrained BL* optimal portfolio" because the former maximizes Sharpe ratio subject to fixed MV inputs and constraints, while the latter simply adjusts τ or the standard deviations of the views, without statistical justification, until the desired sign (or other) inequalities are satisfied. However, the BL+view or BL* means are still derived in part from implied or inverse returns so both procedures are subject to the critiques of that approach.

It is of interest to compare the BL* and classic Markowitz frontiers under the same assumptions. Figure 1 displays the BL unconstrained MV efficient frontier and the associated classical Markowitz frontier for the BL* inputs. Because the Markowitz portfolios are sign constrained, the frontier does not extend as far as the low risk or high return portfolios on the unconstrained BL frontier.[17] However, the two frontiers coincide at the MSR portfolio.

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[13] The value of τ for the BL* portfolio is approximately 0.0702788871. The sign constrained BL optimal portfolio can be computed directly by dividing the view standard deviation, 5% in Table 1, by $\sqrt{\tau}$ = 0.2651016542, resulting in a revised standard deviation of 18.8607%.

[14] As noted earlier, $1/T$ is sometimes proposed as the value of tau in the Black-Litterman formula. For this choice of τ, the BL solution for our example is nearly the same as in Table 2: 20%, 20%, 6%, 0.11%, 1.58%, 6%, 1.58%, 44.73%. For a rigorous data-driven Bayesian analysis, tau represents the factor $1/T$ which multiplies the covariance matrix of an observation to obtain the covariance of the sample mean vector. The procedure is, of course, incorrect for implied returns. Because of the arbitrary scaling of the view standard deviations, setting τ, to $1/T$ does not change any of our conclusions.

[15] These results can be generalized for inequality constrained optimality.

[16] It is worth noting that the formal identity between the BL and Markowitz MSR portfolios with BL* inputs in no way indicates foundational similarity. Unlike BL, the Markowitz MV framework is consistent with standard statistical procedures of inference purely from data.

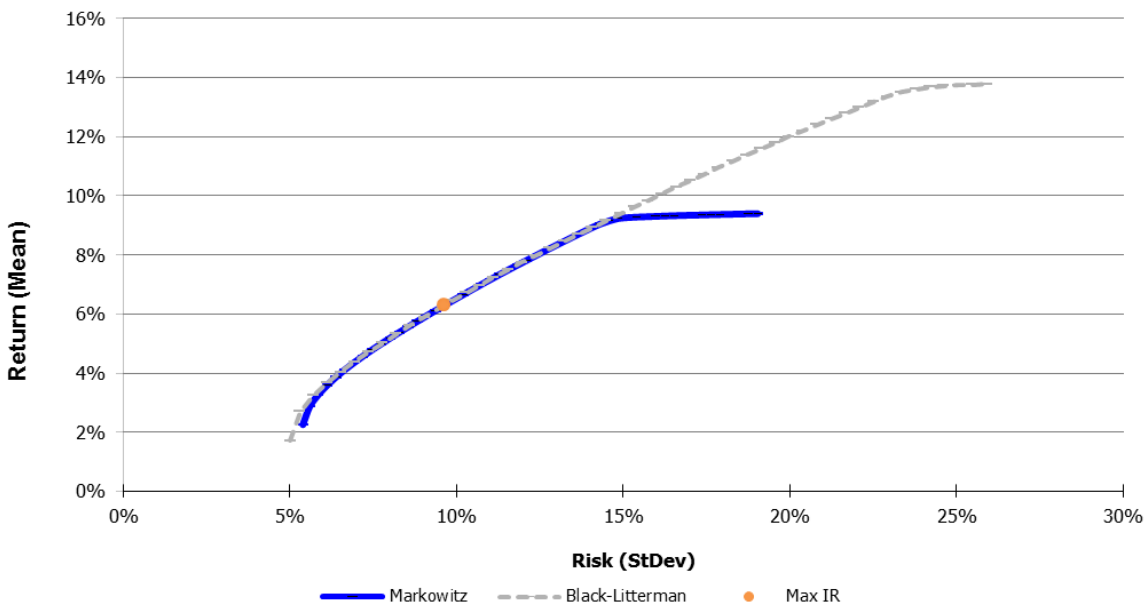
[17] We note that reasonable asset upper bounds on the BL unconstrained efficient frontier were necessary in order that the graph of the frontier would be bounded above.

II.C. Black-Litterman and Michaud Optimization

Michaud optimization uses Monte Carlo resampling methods to address information uncertainty in risk-return estimates in the Markowitz efficient frontier framework. Michaud efficient frontier portfolios are an average of properly associated resampled Markowitz MV efficient frontier portfolios. The procedure is a generalization of the Markowitz efficient frontier conditional on investment information uncertainty.[18]

Table 3 compares the market portfolio in the example with the sign constrained BL* and Michaud optimal portfolios for identical inputs and criteria. Under the same conditions the Michaud MSR portfolio is better diversified, less benchmark centric and less subject to large risky allocations.[19]

Figure 1
Black-Litterman and Markowitz Frontiers
BL* Inputs



*The τ-adjusted means and original covariance are inputs to Markowitz.

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[18] The assumed level of information in risk-return inputs is a parameter of the Michaud optimization process. Different levels of information lead to different efficient frontiers. In this context the Markowitz efficient frontier is a Michaud frontier conditional on 100% certainty. See Michaud and Michaud 2008a, Ch. 6 for further discussion of the patented Forecast Certainty (FC) level parameter. In this case the parameter has been set to match the eighteen years of monthly historical return data in the Michaud dataset.

[19] Rigorous simulation studies have also shown that the Michaud optimized portfolios enhance the out-of-sample investment value of optimized portfolios relative to the MV optimized alternatives (Michaud 1998, Ch. 6; Markowitz and Usmen (MU) (2003), Michaud and Michaud (2008a, b). Harvey et al (HLL) (2008a) present two studies where they dispute the investment enhancement results of Michaud optimization reported in MU. However, Michaud and Michaud (2008c) note that the HLL studies possess significant issues that limit the reliability of their conclusions. Michaud and Michaud (2008c) conclude that HLL's results do not contradict either MU or previous work by Michaud, which they acknowledge in Harvey et al (2008b).

II.D. BL* Markowitz and Michaud Efficient Frontiers

For sign constrained MV optimization, an investor's optimal portfolio is not generally defined by the MSR. Some definition of risk aversion or approximate expected utility may properly define investor optimality relative to efficiently diversified alternatives. Consequently, it is of interest to consider and compare the induced Markowitz and Michaud sign constrained efficient frontier portfolios for this dataset given the BL* inputs. Figure 2 provides a composition map of the optimal asset allocations for the Markowitz efficient frontier portfolios. The horizontal axis represents the risk of the MV optimal portfolios on the efficient frontier. The vertical axis represents the optimal allocations for each asset class at each point on the efficient frontier. The asset classes are color coded for identification and in sequence relative to the asset classes in the tables. For our eight asset class data, the lower risk portfolios on the left hand side of the display show that Euro bonds are a dominant asset while at the extreme expected return or the right hand side of the figure Canada is the dominant asset. Such an exhibit provides a comprehensive demonstration of the dubious diversification and stability inherent in a MV optimization. Note that many asset classes are minimally represented in the BL* input Markowitz efficient frontier portfolios. Note also that efficient frontier allocations may change sharply from one level of risk to another. The MSR Markowitz portfolio with the BL* inputs is identified by the vertical line slicing the composition map at 9.6% portfolio standard deviation.

Table 3
BL* and Michaud MSR Optimized Portfolios

Asset Name	Market	BL*	Michaud
Euro Bonds	20.0%	20.0%	23.0%
US Bonds	20.0%	20.0%	19.9%
Canada	6.0%	6.0%	9.9%
France	6.0%	0.0%	4.3%
Germany	6.0%	1.5%	4.7%
Japan	6.0%	6.0%	6.6%
UK	6.0%	1.5%	5.4%
US	30.0%	45.0%	26.2%
Return	6.1%	5.4%	5.9%
Risk	9.6%	9.5%	9.3%

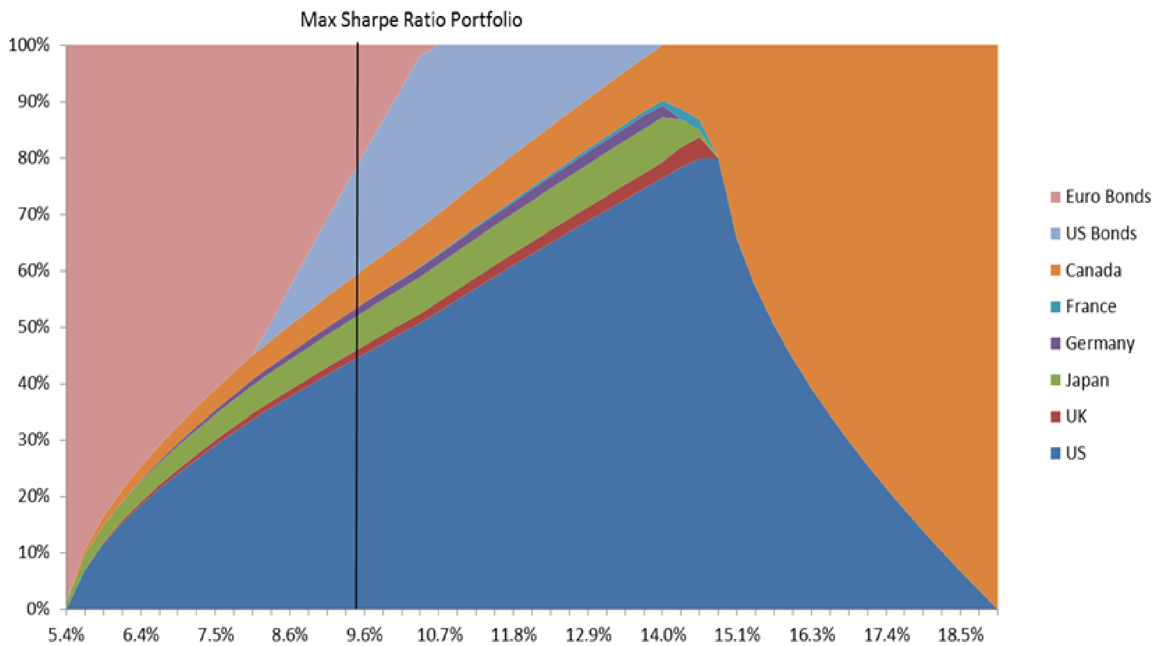
The τ-adjusted means and original covariance of BL are inputs to Michaud, which is computed with 10,000 simulated return series from the t distribution and the arc length rank association frontier-averaging algorithm.

Figure 3 displays the corresponding composition map for the Michaud efficient frontier with BL* inputs. As the display illustrates, the Michaud efficient frontier is characterized by smooth changes in asset allocations from one level of risk to the next. Note that all the assets are well represented in the efficient frontier. The stability of the procedure and diversification indicated at every point on the efficient frontier is of a different order. The Michaud MSR MV optimal portfolio is indicated by a vertical line slicing the composition map.

II.E. Impact of Covariance Estimation Error

BL assumes error-free covariance estimates. However, the assumption is invalid in practice. The covariance matrix is a major source of estimation error for MV optimization. While estimation error accumulates linearly in return it accumulates quadratically in the covariance.[20] As the number of securities increases, all things the same, covariance estimation error may often overwhelm the optimization process.[21]

Figure 2
BL * Inputs Sign Constrained Markowitz Frontier Composition Map



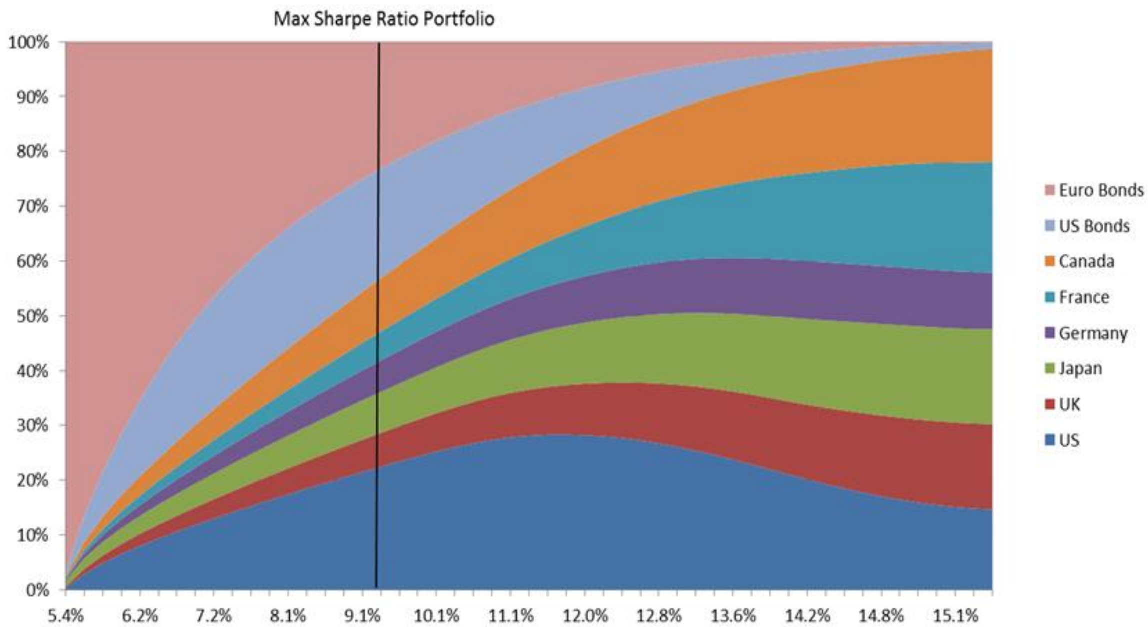
We demonstrate the estimation error sensitivity of the BL optimization process by Monte Carlo simulating the covariance estimates and computing a statistically equivalent range of values of the BL and BL* means. Each simulated covariance matrix produces a statistically equivalent alternative set of covariances and consequently an equivalent set

[20] Noted by Jorion and referenced in Michaud (1998) personal communication.

[21] There is a persistent widespread error associated with the relative importance of estimation error in return relative to risk or the covariance matrix in the professional and academic financial community, usually associated with Chopra and Ziemba (1993). Their analysis is an in-sample and utility function specific study that in no way correctly represents the actual results of the impact of estimation error in rigorous out-of-sample MV optimization simulation tests. As Jorion has noted, estimation error in the covariance matrix may often overwhelm the optimization process as the number of assets increases.

of inverse means.[22] Table 4 displays the BL means from Table 1 in the middle column and the simulated ranges for the indicated assets. For example in Table 1 the BL mean for Canada is 9.2%. The simulated 5th and 95th percentiles for the BL mean values are 7.4% to 11%. Table 5 presents a similar set of results for the BL* means. In Table 1 the BL* mean for Canada is 9.6%. The 5th and 95th percentiles BL* values are 7.1% to 11.8%. We note that the observed variability is greater in Table 5, where the view affects the mean estimates.

Figure 3
BL* Inputs Sign Constrained Michaud Frontier Composition Map



The results show there is no unique set of inverse means when estimation error is considered. Together with the invariance of the implied returns to the data mean, the estimation error sensitivity of the BL procedure is enough to discredit any literal interpretation of BL means as actual estimates of expected return. We note that our estimation error estimates are relatively benign given the generic character of the asset classes and the minimal size of the optimization universe relative to standard institutional applications. In practical settings the consequences of covariance matrix estimation error could be far more detrimental to portfolio performance.

An alternative estimation error computation presents a more dramatic view of BL sensitivity. Assuming the BL means in Table 1 are correct, we simulate statistically equivalent covariance matrices and compute the corresponding implied market

[22] This simulation requires some care. This is because the covariance matrix is devoid of information on return and implied market weights are unique only up to a positive scalar multiple. The simulations in Tables 4 and 5 are normalized so that the average return is always the same as for the original BL means. The assumption of which mean is not material to the simulations. The assumption that the mean is the same limits observed variability.

portfolios. Table 6 provides the results of the experiment. For example, in the case of Euro bonds, the market portfolio weight in Table 1 is 20% while the 5^{th} and 95^{th} percentile simulated asset weights range from -126% to 102%. The results remind the reader that the BL procedure reflects an unconstrained MV optimization framework and confirm that the basic assumptions of BL optimization may be very unreliable even for a simple dataset.

In practice it is essential for effective investment management to require limiting the impact of covariance estimation error. Recommendable approaches include Ledoit and Wolf (2003, 2004) and similar empirical Bayes methods which may limit the impact of highly influential data creating bias in variance and correlation estimates.

Table 4
Ranges of Simulated BL Means

Percentiles:	5%	25%	BL Means	75%	95%
Euro Bonds	1.6	1.9	2.2	2.4	2.8
US Bonds	1.8	2.3	2.6	2.9	3.4
Canada	7.4	8.5	9.2	9.9	11
France	8.4	9.9	10.9	12	13.5
Germany	6.1	7.6	8.6	9.6	11
Japan	4.7	6.5	7.8	9	10.8
UK	8	9.2	10	10.8	12
US	7.4	8	8.5	8.9	9.5

Table 5
Ranges of Simulated BL View Means

Percentiles:	5%	25%	BL View Means	75%	95%
Euro Bonds	1.4	1.9	2.2	2.5	2.9
US Bonds	1.6	2.1	2.6	2.9	3.5
Canada	7.1	8.6	9.6	10.5	11.8
France	3.1	4.5	5.5	6.3	7.5
Germany	1.6	2.9	3.8	4.6	5.8
Japan	1.6	3.4	4.9	6.1	8
UK	4.7	6.2	7.3	8.1	9.4
US	8.2	9.3	10	10.5	11.2

Table 6
Market Portfolio Ranges Assuming BL Means

Market (%)	5%	25%	Market	75%	95%
Euro Bonds	-126	-27	20	58	102
US Bonds	-54	-12	20	58	131
Canada	-12	-1	6	13	27
France	-7	1	6	12	22
Germany	-8	0	6	12	23
Japan	-5	2	6	10	18
UK	-9	0	6	12	24
US	8	20	30	42	67

III. ADDITIONAL BLACK-LITTERMAN ISSUES

III.A. BL is Benchmark-Relative MV Optimization

In BL optimization, the" market" portfolio assumption drives the procedure. However the "market portfolio in equilibrium" is always unknown and more generally indefinable. There is much estimation error of the optimal market portfolio that is ignored, by fiat, by the procedure. The mathematical trick of estimating "implied returns" can make any crazy portfolio MSR optimal. The procedure provides no constraint on investment reality. Essentially, BL optimization solves estimation error in MV optimization by ignoring it.

Benchmarks have often been used to anchor an optimization process and limit ambiguity and instability.[23] However, benchmark centric optimization has serious investment limitations. As Roll (1992) demonstrates, unless the benchmark is precisely ex ante MV efficient, the benchmark centric framework leads to optimization on the wrong efficient frontier. There are always portfolios that have less risk and/or more expected return than the benchmark centric efficient portfolios. Benchmark centricity limits the investment value of BL optimization in practice. [24]

III.B. BL is Unconstrained MV Optimization

Linear inequality constraints fundamental to asset management are ignored in the BL optimization process. Jobson and Korkie (1980, 1981) show that unconstrained MV optimization has little, if any, investment value relative to simple equal weighting. Frost

[23] There may be financially valid reasons for using benchmark centric optimization. One example may be the definition of a benchmark in terms of the "use of invested assets." In this context the benchmark portfolio may represent an individual's or institution's liability. Grinold and Kahn (1995) provide a useful residual return framework suitable for many equity portfolio management mandates in institutional practice. However, their analysis is generally not addressed to asset allocation applications. See Michaud and Michaud (2008a, Chs. 9, 10) for further references.

[24] Roll notes in his conclusion that the suboptimality of the benchmark may need to be balanced against the impact of estimation error on MV optimization.

and Savarino (1988) demonstrate that constraints often reduce the impact of estimation error on MV optimality. Regulatory restrictions and institutional limitations are real world considerations in defining an optimal portfolio. Importantly, Markowitz (2005) shows that consideration of necessary linear constraints in any practical application of portfolio optimization alters the viability of tools of modern portfolio management and important theorems of modern finance. From either a theoretical or practical point of view, proper linear constraints are a necessary condition for effective investment management.

III.C. BL Ignores Investor Risk Aversion

BL optimization computes a single optimal portfolio and does not control for level of investor risk aversion. However, a wide consensus exists in the academic and professional financial community that the choice of portfolio risk is the single most important investment decision.[25] The linear constrained Markowitz and Michaud efficient frontiers provide a range of risk habitats for rational investment decision making. The BL portfolio risk level may often be inappropriate for many investors. Efficient frontiers are essential for managing investor risk habitats.

IV. SUMMARY AND CONCLUSION

Black-Litterman (1992) (BL) optimization produces a single maximum Sharpe ratio (MSR) optimal portfolio on the unconstrained MV efficient frontier based on an assumed MSR optimal benchmark portfolio and active views. BL optimization often results in uninvestable portfolios in applications due to large leveraged and/or short allocations. BL introduce an input tuning parameter τ to compute BL* sign constrained portfolios. We present a mathematical and statistical analysis of BL optimization and illustrate the procedure with a simple dataset of eight asset classes. We compare constrained BL to Markowitz and Michaud under identical conditions.

BL optimization claims to solve the investment limitations in practice associated with classical Markowitz MV optimization. However, we show that the ad hoc "market" portfolio assumption drives the process and that estimation error associated with estimating the "market portfolio in equilibrium" is solved by ignoring it. In addition the unconstrained MV optimization framework used by BL is known to have little, if any, investment value, and constrained BL optimization is often identical to Markowitz and consequently inherits the known limitations of MV optimization. Moreover, the process of computing "inverse" returns and adjusting the prior with τ-adjustment is inconsistent with principles of modern statistical inference and rigorous Bayesian analysis. Finally, the BL optimal solution ignores the importance of investor risk aversion while the resampling

[25] Brinson et al (1986, 1991).

process associated with Michaud optimization produces superior risk managed and diversified portfolios.

Portfolio "acceptability" is more the norm for asset management in practice than widely recognized. Many investment firms do not use optimization technology and their investment process closely mirrors the BL framework of positing a benchmark and considering investment tilts. [26] Even in a more formal optimization investment process, instability and unintuitiveness often motivate ad hoc practices that amount to little more than the computation of a committee's notion of "acceptable" portfolios. However, acceptable is not investment effective. Effective asset management requires an inequality constrained optimization framework, an efficient frontier of optimal risk managed portfolios for satisfying risk habitats, input estimation consistent with modern statistical inference and estimation error effective portfolio optimization. While BL may be convenient it is not recommendable given available alternatives. The potential for adding value with rigorous investment and statistical principles seems a challenge well worth the effort.

APPENDIX

Table 1A

Correlations

Asset Name	Euro Bonds	US Bonds	Canada	France	Germany	Japan	UK	US
Euro Bonds	1	0.92	0.33	0.26	0.28	0.16	0.29	0.42
US Bonds	0.92	1	0.26	0.22	0.27	0.14	0.25	0.36
Canada	0.33	0.26	1	0.41	0.30	0.25	0.58	0.71
France	0.26	0.22	0.41	1	0.62	0.42	0.54	0.44
Germany	0.28	0.27	0.30	0.62	1	0.35	0.48	0.34
Japan	0.16	0.14	0.25	0.42	0.35	1	0.40	0.22
UK	0.29	0.25	0.58	0.54	0.48	0.40	1	0.56
US	0.42	0.36	0.71	0.44	0.34	0.22	0.56	1

[26] The process is described in Michaud (1989).

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