Optimal Multiperiod Mean-Variance Portfolio Growth Investment Policy

by

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ABSTRACT

The problem of defining an optimal multiperiod investment policy with respect to the mean-variance (Hakansson) efficient frontier of the portfolio's growth rate is examined. Dynamic programming computer solutions are derived that define the optimal percent of assets to be placed at risk, or choice of portfolio beta, in each period of a two period investment horizon. The optimal solutions possess intuitively appealing properties that are consistent with some traditional investment management practices. Comparisons between optimal and rebalanced investment policies illuminate some deficiencies in non-optimal investment policy. In particular, even though no ability to time the market is assumed, holding a portfolio's asset allocation or beta constant period-by-period is unlikely to be an optimal use of the "time option" implicit in investment management over time.

Empirical evidence on the behavior of capital markets (Fama, 1970; Jensen, 1972) suggests that the single most important investment decision is probably the assumed level of systematic risk (beta). From a more practical point of view, nearly 94% of the variance of annualized return of large institutional pension funds can be explained by the plan's average asset mix (Brinson and Diermeier, 1985). It is therefore of interest to consider the problem of the efficient management of systematic risk or defining an efficient asset allocation investment policy over time. Since the geometric mean (compound return, growth rate) is the appropriate statistical measure of return over time, a natural multiperiod generalization of the Markowitz (1959) mean-variance efficient frontier portfolio selection criterion is to select portfolios in accordance with the mean-variance efficient frontier of the geometric mean distribution. Such a criterion avoids the important defect of the mean-variance single-period models that continued reinvestment over time may lead to ruin with probability one (Hakansson, 1971a), and allows a choice of a suitable level of portfolio risk.

Numerous authors have examined the investment properties of the almost sure limit of geometric mean return as the number of periods $N \rightarrow \infty$ as a portfolio selection objective.[1] In particular, this criterion provides a simple prescriptive rule in each period ($\max E(\ln(1 + r))$) and leads, over a sufficiently long investment horizon, almost surely to more wealth than any other essentially different investment strategy (Thorp, 1974). More generally, Hakansson (1971a,b) and Hakansson and Miller (1975) have examined some investment characteristics of the mean-variance geometric mean "Hakansson" efficient frontier and have proposed the criterion as a useful investment objective for managing multiperiod portfolio return.

Some important objections have been raised to the maximum expected geometric mean portfolio objective. In particular, Samuelson and Merton (1974) have shown that it is not consistent with expected utility maximization under a wide variety of conditions. The Samuelson-Merton objections are fundamentally concerned with the fallacy of attending to (any of) the statistical parameters of the return or terminal wealth distribution as surrogates for the mean of the utility of terminal wealth. Their objections apply broadly to most investment management tools in current institutional use. This is because, as a practical matter, financially valid and relevant utility functions are almost never available as a portfolio selection criterion. As a consequence, optimisation of various statistical characteristics of the return distribution is a common generic principle underlying most, if not all, practical investment management techniques.

It is beyond the scope of this paper to investigate the foundations of investment criteria for institutional portfolio management.[2] From the perspective of practical portfolio management, the expected value of the portfolio growth rate over an investment horizon is often an important part of stated investment objectives and ex post performance evaluation of professionally managed portfolios. An operational and valuable method for the resolution of such controversies is to explore the investment characteristics of a proposed portfolio selection criterion. Used with an awareness of its characteristics and limitations, the Hakansson criterion may be a useful benchmark and practical guide for the investment management decision making process for investors with investment objectives consistent with properties of the criterion.

We consider the following simple but fundamental investment problem. How should an investment manager optimally divide assets to be placed at risk in each time period or "play" of the investment "game," where the remainder of assets is invested in a riskless asset. We assume that the return distribution is identically distributed and intertemporally independent over time.

As a convenience, we will use the term "beta" to describe the investment policy decision parameter, the proportion of assets to be placed at risk in each period. To simplify the analysis, we will also assume that the investor can borrow at the riskless rate.

An equivalent formulation can be described if we assume a perfectly diversified portfolio in perfect and efficient capital markets in the context of a return generation process that is consistent with the Security Market Line (SML) of the Sharpe (1864)-Lintner (1965) Capital Asset Pricing Model (CAPM) or the Ross (1977) single-period single-factor model in each period. In this case, "beta" represents the more traditional notion of a measure of normalized covariance with the return of the "market" portfolio or single factor.

Since many institutional portfolios have a constant beta investment policy, we will want to compare optimal to "rebalanced" investment policies. Michaud (1981) investigated the multiperiod mean-variance geometric mean consequences of following a rebalanced beta policy over a given N-period investment horizon. The results of our analysis will show that an optimal multiperiod beta policy in the context of the Hakansson criterion will not generally be constant over time.[3] A practical implication of the results suggest that, even in the absence of market timing information, a constant beta investment policy ignores an important "time option" that may lead to significantly suboptimal investment performance over time.

Solution for the Hakansson efficient frontier beta investment policies is a dynamic programming problem. Because of computational difficulties, we examine the simplest non-trivial version of the problem: a two-period investment horizon with a two-point return distribution in each period. While obviously limited in scope and applicability, these simple cases nevertheless appear to provide valuable information on the general structure of the Hakansson multiperiod investment process. The optimal policies possess intuitively appealing characteristics that provide a rationale for some traditional investment management practices. Similar in certain significant respects to the effect of single-period portfolio diversification, the criterion provides a kind of "multiperiod diversification" that is most likely to appeal to institutional investors who may be willing to trade off a diminished probability of large positive returns for an improvement in typical fund performance over time.

Section I describes the Hakansson efficient frontier criterion and its dynamic programming characteristics. Section II provides the computer solutions to the Hakansson efficient frontier optimal beta policies for the two-period two-point return distribution case. Section III discusses the investment characteristics and implications of the optimal beta policies. Section IV examines some limitations of the analysis. Section V provides a summary.

PROBLEM FORMULATION

The N-period geometric mean return is defined as:

$$(1) \qquad G_N=((1+r_1)\ (1+r_2)\ \ldots\ (1+r_N))^{1/N} - 1$$

where $r_i > -1$, $i=1, \ldots, N$, is the single-period return in the ith period of the N-period investment horizon. We assume that cash flows are absent and returns reinvested. The N-period terminal wealth ratio is defined as:

$$(2) \qquad W_N= \prod^{N} (1+r_i)\ .$$

In each period we assume that portfolio total return, Rp, is generated according to the linear relationship

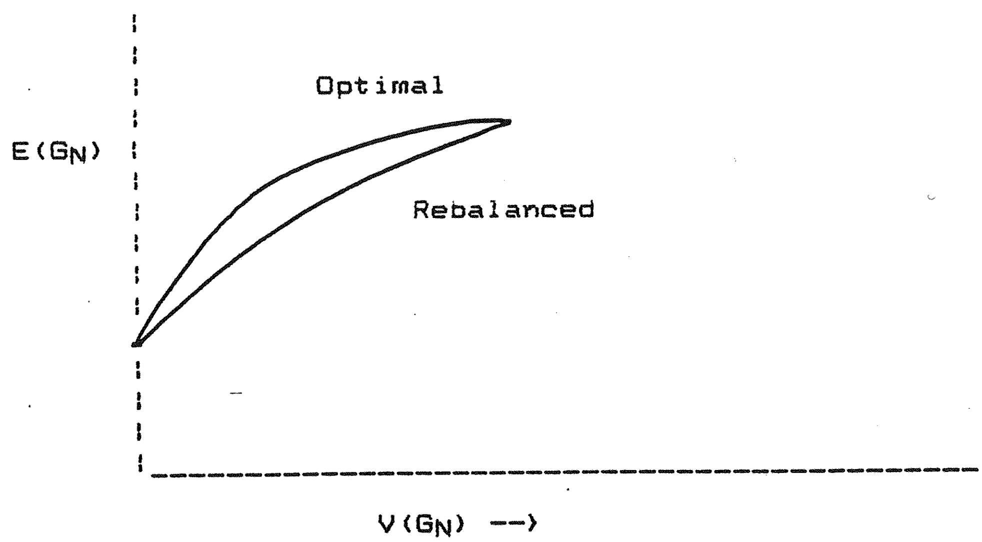
$$(3) \qquad Rp= R_0 + Bp(R - R_0)$$

where R is the total return on the risky asset, R_0 is the riskless rate and Bp is the portfolio "beta"; i.e., the proportion of assets placed at risk or the level of systematic risk of the portfolio defined by

(4) $Bp = cov(Rp, R)/var(R)$.

In intuitive terms, given an outlook for the market environment over a given N-period investment horizon, the manager seeks a rule prescribing that percent of assets to be dedicated to the risky asset in each period that will provide the largest expected portfolio growth rate for a given level of the growth rate risk or lowest growth rate risk for a given expected portfolio growth rate over the entire investment horizon. Our objective is to compare the optimal solutions to rebalanced policies with the same risk or expected return. Comparison of rebalanced versus Hakansson efficient investment policies over an N-period investment horizon in the context of the Hakansson efficient frontier is illustrated in Figure 1. Note that no period-by-period "market timing" expertise is assumed to be available to the portfolio manager in our problem formulation.

FIGURE 1
HAKANSSON EFFICIENT FRONTIER
OPTIMAL VS REBALANCED INVESTMENT POLICIES



Hakansson (1971b) has shown that optimal investment policies are rebalanced policies for two cases: 1) zero risk, where it is assumed that a zero-risk asset exists for the investment horizon; 2) the maximum expected geometric mean. These cases are illustrated in Figure 1 by the coincidence of the frontiers at the extremes. Consequently, optimal and rebalanced multiperiod investment policies will not differ significantly either in their characteristics or consequences near the extreme points of the frontier. In other cases, an optimal investment policy with respect to the Hakansson criterion will generally be variable over the period.

The general solution of an N-period optimal investment policy over time associated with the Hakansson criterion is a dynamic programming problem which can be written in two alternative and equivalent forms:

(5) Maximize
$$\emptyset = E(G_N) + \lambda(\sigma^2(G_N) - \sigma^2_s)$$

or

(6) Minimize
$$\emptyset = \sigma^2(G_N) + \lambda(E(G_N) - E_s)$$

where

σ^2_s = geometric mean variance of the rebalanced or stationary investment policy

E_s = expected geometric mean of the rebalanced investment policy

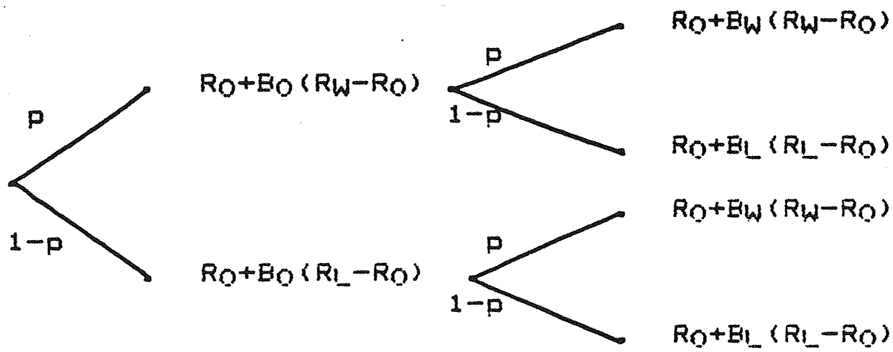
λ = the Lagrangian multiplier.

In the case of (5) we are solving for investment strategies that maximize the expected growth rate or geometric mean for a given level of the variance over the investment horizon. Similarly, (6) solves for investment policies which minimize the variance of the growth rate for a given level of the expected return. Since the characteristics of the formulations (5) and (6) are entirely equivalent, our results will focus exclusively on the solutions of (5).

For our problem, the mean and variance of the geometric mean are functions solely of the values of beta chosen in each period. The optimal beta strategies, in each period beyond the first, are conditionally defined functions which depend on the history of returns experienced by the portfolio manager for the previous periods. For the two-period two-point distribution case, there are three possible values of beta. In the first period there is no prior return history; therefore, the definition of an optimal beta is unconditional and is denoted by B_o* In the second period, the number of optimal betas depends on the number of possible returns in the first period. In our case, we denote R_W, the favorable return or "winning" outcome for the risky asset and R_L, the unfavorable return or "losing" outcome. Corresponding to each return of the risky asset the portfolio manager should optimally choose a value of beta which reflects his return experience; either B_W*, corresponding to R_W or B_L*, corresponding to R_L.

Schematically, the two-period two-point return distribution dynamic programming problem for the assumed one-factor model of single-period return and risk free rate can be described in terms of a tree diagram given in Figure 2.

FIGURE 2
DYNAMIC PROGRAMMING FORMULATION
TWO-PERIOD TWO-POINT RETURN DISTRIBUTION
ONE FACTOR MODEL, PERFECT DIVERSIFICATION



The mathematical structure of the dynamic programming problem can be described in terms of a non-linear system of four equations in four unknowns defined by setting the partial derivatives of Ø to zero with respect to B_O, B_W, B_L, and λ. A generalized Newton-Raphson technique was used to solve iteratively for the four unknowns in the non-linear system of four equations[4]. All computations were performed on a CDC Cyber 71 computer.

COMPUTATIONAL SOLUTIONS

Tables 1-3 summarize the results of the dynamic programming computer solutions of (5) that compare Hakansson efficient frontier two-period optimal beta policies to rebalanced beta policies that have the same two-period geometric mean return standard deviation. The optimal beta solutions are derived with respect to three rebalanced beta policies: 0.5, 1.0, 1.5, and for three single-period two-point distributions of risky asset returns: 20% and 0%, 30% and -10%, 40% and -20%. Table 1 represents the case where the assumed two-point risky asset return distribution is symmetric about its mean (p=.5 for both outcomes). Tables 2 and 3 represent the case where the two-point return distribution is asymmetric; Table 2 represents a right-skewed return distribution (p=.45 for the favorable return outcome), Table 3 represents a left-skewed return distribution (p=.55 for the favorable return outcome).

Since Tables 1-3 are fundamentally similar, only the data in Table 1 will be described in detail. In this case, the risk free rate and probability of favorable outcome, p, is given as 5% and .5 respectively. There are three rebalanced beta policies that will be compared to their Hakansson optimal counterparts, B_S, equal to 0.5, 1.0, and 1.5 which define the three major groups of data in the columns of the table. Within each rebalanced beta case, there are three return distribution cases examined, indicated in the column headings for the row R. For each column defined by B_S and R, the data for the optimal solutions is given. The first three rows describe the Hakansson optimal beta policies. For example, the row of data, $B_O{}^*$ describes the optimal beta policy for the first period of the two period investment horizon. This result should be compared with the corresponding rebalanced beta policy B_S at the top of the table, in this case 0.5. The following two rows describe the Hakansson optimal beta policies conditional on the return of the risky asset in the first period.

The following rows provide information on either the mean and standard deviation of the growth rate or the mean, standard deviation and median of the terminal wealth distribution. For example, the row following the beta policies describes the expected geometric mean for the rebalanced policy and can be compared to the subsequent row which describes the same statistic for the optimal policy. As we have defined the problem, both the optimal and rebalanced policy has the same standard deviation, which is given in the following row without superscript or subscript. The remaining rows describe respectively the mean and standard deviation of rebalanced terminal wealth, mean and standard deviation of optimal terminal wealth and comparison of the medians of the rebalanced versus the optimal terminal wealth distributions.

Table 1 shows that if the single-period mean return is held constant while the single-period variance is increased (20%,0% to 40%,-20% risky asset returns), the expected geometric mean decreases and the standard deviation increases. This fundamental principle is at work throughout all three tables. In Tables 2 and 3, changing the probability of the favorable outcome changes the single-period standard deviation and mean. For the data in the three tables, the primary determinant of the characteristics of the optimal solutions is the ratio of the single-period risk premium to standard deviation.

Generally, as the value of the rebalanced beta is increased, the expected geometric mean is increased. However, a rebalanced 1.5 beta policy for the 40%,-20% market return distribution case in Table 1 represents a point that has a smaller expected geometric mean and larger standard deviation than the rebalanced beta policy at 1.0. The 1.5 rebalanced beta policy represents a point beyond the maximum point on the Hakansson efficient frontier and therefore has no optimal solution on the Hakansson efficient frontier. This situation corresponds to a rebalanced beta policy larger than the "critical" beta as described in Michaud (1981).

Table 4, which is similar in format to the previous tables, further refines the analysis of the data. It provides each two-period terminal wealth ratio (percent change) outcome for the rebalanced (W_S) and Hakansson efficient (W^*) beta policies. The columns of data correspond to the indicated values of B_S for the indicated values of the risky asset return distribution. In the first column, each three sets of numbers represent the smallest, middle and largest terminal wealth outcomes for either the rebalanced beta case, denoted by W_S, or the optimal beta cases, denoted by W^*, further subdivided by the probability of the favorable outcome. To the level of numerical significance given, there are always three two-period wealth outcomes in each case. The middle outcome is either exactly or nearly twice as probable as the other two outcomes.

CHARACTERISTICS OF AN OPTIMAL MULTIPERIOD INVESTMENT POLICY

Except at the extremes of the Hakansson efficient frontier, the following inequalities summarize the results concerning the Hakansson optimal investment policies:

$$B_L^* > B_O^* > B_S > B_W^* \quad (7)$$

SETTING INITIAL POLICY: The inner inequality in (7) implies that the optimal initial portfolio risk policy requires taking "higher than normal" risk with respect to target or rebalanced policy. The portfolio manager then relies on the availability of the "time option" over ensuing periods to adjust over all risk consistent with investment experience in previous periods.

ADJUSTING POLICY TO REFLECT INVESTMENT EXPERIENCE: The results in (7) show that the Hakansson optimal second period beta policy is to reduce the level of portfolio risk given favorable initial period returns and increase the risk level in the second period given unfavorable initial period returns. This prescription corresponds to often quoted traditional investment management wisdom: When the portfolio manager is "ahead" he should "trim his sails;" correspondingly, if he is "behind" he should put "sail to the wind." Of course, the dynamic programming solutions also provide quantitative estimates, under the assumptions, concerning the magnitude and asymmetry of the appropriate risk level adjustments.

THE EFFECT OF THE INVESTMENT ENVIRONMENT ON OPTIMAL POLICY: As the variance of the two-point return distributions increases, (e.g., comparing the 20%,0% to 40%,-20% return distribution columns in the tables) or as the mean diminishes (e.g. corresponding lines in Tables 1 and 2, the optimal risk adjustments approach the rebalanced beta policy. Consequently, in market environments with high risk or low expected return, optimal beta policies should deviate little from the rebalanced beta policy. Conversely, in market environments with low levels of uncertainty or high expected return, optimal beta policies may deviate significantly from the rebalanced policy. The Hakansson criterion implies that the optimal management of systematic risk over time should be "active" in a stable and/or high return environment and "steady" in a highly uncertain and/or low return one.

THE EFFECT OF OPTIMAL INVESTMENT POLICY: Parameters of the two-period geometric mean and terminal wealth distributions provide a means for the evaluation of the investment characteristics of Hakansson optimal beta policies. Except at the extremes of the frontier, the following relationships were observed:

(8) $E^*(W) > E_s(W)$

(9) $\sigma^*(W) < \sigma_s(W)$

(10) $M^*(W) > M_s(W)$

(11) $\max^*(W) < \max_s(W)$

(12) $\min^*(W) < \min_s(W)$

ENHANCING GEOMETRIC MEAN RETURN: We note that the expected geometric mean return is enhanced when an optimal investment policy is used, in contrast to a rebalanced policy, but that the enhancement is small.

ENHANCING TYPICAL INVESTMENT PERFORMANCE: The major effect of an optimal investment policy is primarily revealed in the properties of the terminal wealth distribution. Of particular investment interest is that the optimal beta policies lead to a substantial increase in median or middle outcome in the two-period terminal wealth distribution. However, the Hakansson optimal beta policies also reduce the largest and smallest outcomes available with respect to the rebalanced beta terminal wealth distribution. Based on the computer solutions, Hakansson optimal investment policies shift the center of the distribution to the right while shifting the extremes to the left without decreasing the mean and decreasing the standard deviation.

Hakansson optimal investment policy represents a kind of "multiperiod diversification." The empirical effect of single-period diversification with respect to the underlying distribution of individual stock returns is to increase median return at the cost of lowering the probability of achieving higher returns (Fisher and Lorie, 1970). The most important difference is that the Hakansson optimal policy increases, not diminishes, the probability of lower returns. The Hakansson investor should be willing to trade off the possibility of large gains, and an increased probability of small gains, for improvement in typical performance of the fund over the investment horizon.

THE EFFECT OF LONGER TIME PERIODS ON RETURN ENHANCEMENT POTENTIAL: Some characteristics of Hakansson optimal investment policies for longer time horizons can be anticipated. As the number of periods increase, the geometric mean standard deviation (for rebalanced or optimal beta policies) will decrease and approach zero while the maximum expected geometric mean is non-increasing (generally decreasing) as a function of the number of time periods and approaches a well defined limit ($e^{E(\ln(1+r)}-1$) (Michaud, 1981). Therefore, referring to Figure 1, the risk reduction differential of Hakansson optimal beta policies (horizontal difference between the optimal and rebalanced curves) will decrease as the number of periods in the investment horizon increases. However, the expected geometric mean return enhancement differential of optimal beta policies (vertical distance between the curves) need not diminish significantly as the number of investment periods increases.

SOME CAVEATS AND LIMITATIONS

APPLICATIONS TO ACTUAL INVESTMENT MANAGEMENT: The problem we have examined was intended as a simple first step in understanding the structure of optimal multiperiod investment policy. Perhaps the most important limitations of the problem reflect the two-period investment horizon assumption and the lack of transaction costs. We will discuss each in turn.

THE TWO-PERIOD INVESTMENT HORIZON: There are many investment situations where the investment setting, such as a mutual fund or pension fund, requires a long term investment horizon. There are, however, many cases, such as that of a pension fund portfolio manager, where the "account life" may have a relatively short horizon and where performance is monitored over shorter, usually quarterly or semi-annual, subperiods. In the latter case, our results may have some fairly direct applications. In the former case, our results may provide some useful guidelines for better understanding of the use of the "time option" for managing investment policy.

It seems reasonable to speculate that when the number of periods in the investment horizon increases and/or when the length of each time period shortens, the prescribed period-by-period adjustments of beta may require less substantive changes. While the limitation of a fixed investment horizon is problematical, its practical effects may diminish in the context of more realistic horizons; the fixed horizon may be less important then the period-by-period prescription of optimal management of systematic risk to achieve overall, but perhaps distant, superior portfolio growth.

TRANSACTION COSTS: Hakansson optimal beta policies imply more buying and selling of risky assets on a relative basis than a rebalanced policy. Such increased transaction costs may mitigate some or all of the benefit of the optimal policy. It is of interest to note that the zero transaction cost case, a buy-and-hold policy, cannot be compared to rebalanced or optimal multiperiod investment policies. This is because a buy-and-hold policy does not control for portfolio growth rate risk; some transaction costs are required of any multiperiod investment policy that controls portfolio risk.

One alternative is to reduce the level of transaction costs associated with required changes of systematic risk level by using stock index futures or options on stock index futures (see, e.g., Zeckhauser and Niederhoffer, 1983). However, if the investment horizon includes more subperiods, the transaction costs differential between an optimal and rebalanced investment policy, may be less significant, when related to expected benefits. This is because, the optimal beta policies may not deviate as substantially on a period-by-period basis as they do in the two-period horizon case and because even a small increase in the growth rate of the portfolio over relatively long periods of time may lead to substantive increases in terminal wealth.

COMPARISON WITH A "PORTFOLIO INSURANCE" INVESTMENT POLICY: It is of some interest to compare a Hakansson optimal investment policy and that which follows from a "portfolio insurance" dynamic hedging strategy.[5] Ignoring the hedge ratio, the recommended portfolio insurance beta policy is the inverse of the Hakansson policy: buy when returns were favorable and sell when returns were unfavorable. The consequences of an "inverse" Hakansson efficient frontier policy will result in lowering the probability of low returns and increasing the probability of high returns by lowering typical fund performance. Also, an "inverse" Hakansson policy may result in a relative increase in transaction costs.

This is because, on a relative basis, it may be less costly to buy (sell) assets that have recently performed poorly (well), as in the Hakansson policy case, than to buy (sell) assets when they have recently performed well (poorly), as in the portfolio insurance case.

SUMMARY

The problem of setting an optimal level of systematic risk or investment policy beta over time in the context of the Hakansson efficient frontier criterion was discussed. Computer solutions for simple two-period two-point return distributions were given. The structural character of the optimal beta strategy was compared to a rebalanced beta policy with the same variance of the geometric mean. For the cases examined, the optimal initial period beta is to take "more than normal" risk with respect to the corresponding rebalanced portfolio beta taking advantage of the "time option" to control portfolio return in a multiperiod setting. The optimal (conditional) beta policies in the second period were consistent with the often quoted investment management wisdom of "trimming your sails" given favorable investment experience and "putting sail to the wind" given unfavorable investment experience. The results also imply that optimal beta management over time should be "active" in a favorable market environment and "steady" in an unfavorable one.

A Hakansson optimal investment process can be roughly described as providing "multiperiod diversification." The major benefit can be seen from a significant improvement in median terminal wealth. The optimal policies also result in a decreased probability of higher returns and an increased probability of lower returns. While the Hakansson investment policies also improve the expected geometric mean with respect to a corresponding rebalanced beta policy, the improvement is small. Although Hakansson efficient frontier solutions possess attractive multiperiod risk management and terminal wealth distribution characteristics, the criterion appears to be primarily useful to investors whose investment objectives are associated with improving typical investment performance.

There are fundamental limitations of the simple problem of the analysis that includes the assumption of a fixed, two-period investment horizon and the lack of explicit consideration of relative transactions costs with respect to the rebalanced investment policy. Some suggestions were offered to indicate that the practical effect of these limitations may not be substantial in the context of longer time horizons, shorter individual periods, and the use of strategies that minimize transaction costs. In fact, the surprising result is that such a simple problem appears to have so many interesting implications for the structure of optimal multiperiod investment management. The results provide a guide and should be viewed as a simple first step towards better understanding of the management of multiperiod investment policy. One apparently reliable and important conclusion relates to the likely non-optimality of a fixed beta investment policy.

Finally, the astute reader may have noticed that a Hakansson optimal investment policy is also a description of an optimal gambling strategy over time. It is, however, well known that an optimal gambling strategy for a fair game does not exist. The critical difference is that the investment "game" is assumed to have a positive expected risk premium, while in most games of chance the risk premium is zero or negative. In the context of our problem it is easy to show that when the expected risk premium is zero or less, the optimal gambling strategy will not deviate from a rebalanced one. This is because, under these conditions, referring back to Figure 1, the Hakansson efficient frontier is a point. Consequently, the optimal gambling strategy is to have zero percent of assets at risk or "gambled" at each play of the game. Our results have non-trivial risk management content only because we have assumed a positive premium for bearing risk.

TABLE 1
HAKANSSON EFFICIENT AND REBALANCED BETA POLICIES
TWO-PERIOD GEOMETRIC MEAN AND TERMINAL WEALTH STATISTICS
SYMMETRIC SINGLE-PERIOD RETURN DISTRIBUTION
$p=.5$, $R_0=5\%$

B_S	.5			1.0			1.5		
R	20,0	30,-10	40,-20	20,0	30,-10	40,-20	20,0	30,-10	40,-20
B_0^*	.59	.52	.50	1.18	1.03	1.00	1.76	1.53	1.50#
B_W^*	.24	.40	.45	.51	.85	.98	.80	1.35	1.54
B_L^*	.72	.61	.55	1.45	1.18	1.03	2.20	1.71	1.42
$E_S(G)$	7.4	7.3	7.0	9.8	9.1	7.9	12.0	10.5	7.8
$E^*(G)$	7.6	7.3	7.0	10.0	9.1	7.9	12.4	10.5	7.8
$\sigma(G)$	3.5	7.1	10.6	7.1	14.2	21.3	10.6	21.3	32.2
$E_S(W)$	15.6	15.6	15.6	21.0	21.0	21.0	26.6	26.6	26.6
$\sigma_S(W)$	7.6	15.2	22.9	15.6	31.4	47.5	24.0	48.6	74.4
$E^*(W)$	15.9	15.6	15.6	21.2	21.1	21.0	27.4	26.6	26.6
$\sigma^*(W)$	7.5	15.0	22.7	15.2	30.7	47.3	23.0	47.5	75.1
$M_S(W)$	15.3	14.6	13.3	20.0	17.0	12.0	24.3	17.6	6.3
$M^*(W)$	18.1	16.8	14.9	25.7	20.6	12.8	32.8	21.3	4.7

This column denotes a case not on the Hakansson efficient frontier.

TABLE 2
HAKANSSON EFFICIENT AND REBALANCED BETA POLICIES
TWO-PERIOD GEOMETRIC MEAN AND TERMINAL WEALTH STATISTICS
RIGHT SKEWED SINGLE-PERIOD RETURN DISTRIBUTION
$p=.45$, $R_0=5\%$

B_S		.5			1.0			1.5	
R	20,0	30,-10	40,-20	20,0	30,-10	40,-20	20,0	30,-10	40,-20
B_0^*	.56	.51	.50	1.12	1.01	1.00#	1.67	1.50	1.50#
B_W^*	.28	.44	.50	.58	.94	1.06	.90	1.48	1.64
B_L^*	.66	.55	.50	1.34	1.07	.92	2.01	1.53	1.26
$E_S(G)$	6.9	6.3	5.5	8.8	7.1	4.9	10.5	7.5	3.4
$E^*(G)$	7.0	6.3	5.5	8.9	7.1	4.9	10.7	7.5	3.4
$\sigma(G)$	3.5	7.0	10.5	7.0	14.0	20.9	10.5	20.9	31.4
$E_S(W)$	14.5	13.4	12.4	18.8	16.6	14.5	23.2	19.9	16.6
$\sigma_S(W)$	7.6	15.0	22.5	15.4	30.7	46.0	23.5	47.1	71.3
$E^*(W)$	14.5	13.4	12.4	19.1	16.6	14.5	23.6	19.9	16.6
$\sigma^*(W)$	7.4	14.9	22.5	15.0	30.4	46.6	22.8	46.9	73.5
$M_S(W)$	15.3	14.6	13.3	20.0	17.0	12.0	24.3	17.6	6.3
$M^*(W)$	17.5	15.7	13.3	24.3	18.4	9.8	30.7	18.1	0.8

This column denotes a case not on the Hakansson efficient frontier.

TABLE 3
HAKANSSON EFFICIENT AND REBALANCED BETA POLICIES
TWO-PERIOD GEOMETRIC MEAN AND TERMINAL WEALTH STATISTICS
LEFT SKEWED SINGLE-PERIOD RETURN DISTRIBUTION
$p=.55$, $R_0=5\%$

B_S	.5			1.0			1.5		
R	20,0	30,-10	40,-20	20,0	30,-10	40,-20	20,0	30,-10	40,-20
B_0^*	.63	.54	.52	1.26	1.06	1.02	1.88	1.57	1.51
B_W^*	.22	.36	.42	.46	.78	.91	.71	1.25	1.45
B_L^*	.78	.67	.62	1.59	1.32	1.16	2.43	1.92	1.60
$E_S(G)$	7.9	8.3	8.5	10.8	11.1	10.9	13.5	13.5	12.4
$E^*(G)$	8.2	8.4	8.5	11.3	11.2	11.0	14.2	13.7	12.4
$\sigma(G)$	3.5	7.1	10.6	7.1	14.2	21.5	10.6	21.5	32.6
$E_S(W)$	16.6	17.7	18.8	23.2	25.4	27.7	30.0	33.4	36.9
$\sigma_S(W)$	7.6	15.3	23.1	15.6	31.8	48.5	24.2	49.6	76.8
$E^*(W)$	17.2	17.9	18.9	24.3	25.7	27.8	31.5	33.7	36.9
$\sigma^*(W)$	7.5	15.0	22.7	15.2	30.7	47.8	23.1	47.7	75.9
$M_S(W)$	15.3	14.6	13.5	20.0	17.0	12.0	24.3	17.6	6.3
$M^*(W)$	18.9	18.0	16.5	27.2	22.8	15.8	35.2	24.5	8.4

TABLE 4

HAKANSSON EFFICIENT AND REBALANCED BETA POLICIES
TWO-PERIOD TERMINAL WEALTH DISTRIBUTION POLICIES

B_S	.5			1.0			1.5		
R	20,0	30,-10	40,-20	20,0	30,-10	40,-20	20,0	30,-10	40,-20
W_S	5.1	-4.9	-14.4	0.0	-19.0	-36.0	-4.9	-31.9	-54.4#
	15.3	14.6	13.3	20.0	17.0	12.0	24.3	17.6	6.3
	26.6	38.1	50.1	44.0	69.0	96.0	62.6	103.	148.
					p=0.5				
W*	3.5	-6.8	-15.8	-3.1	-21.1	-36.7	-9.4	-34.9	-53.1#
	18.1	16.8	14.9	25.7	20.6	12.8	32.8	21.3	4.7
	23.7	35.6	48.3	38.2	64.9	95.0	53.7	98.7	150.
					p=.45				
W*	3.9	-5.8	-14.4	-2.2	-20.0	-34.4#	-8.2	-32.3	-50.5#
	17.5	15.7	13.3	24.3	18.4	9.8	30.7	18.1	.8
	23.7	36.5	50.1	38.3	67.1	98.8	54.0	102.	156.
					p=.55				
W*	3.0	-8.0	-17.5	-4.2	-24.1	-39.5	-11.2	-37.9	-56.3
	18.9	18.0	16.5	27.2	22.8	15.8	35.2	24.5	8.4
	23.9	35.0	47.2	38.5	63.7	92.4	54.2	96.6	146.

This case represents a point not on the Hakansson efficient frontier.

FOOTNOTES

1 Breiman, 1960; Hakansson, 1971b; Kelly, 1956; Latane, 1959; Markowitz, 1959, Ch. 6; Thorp, 1974.

2 Hakansson, 1979; Levy and Markowitz, 1979; Merton and Samuelson, 1971; Samuelson and Merton, 1974.

3 The derived computer solutions in Tables 1-3 provide counter-examples to an assertion made by Hakansson (1979, p. 174, A5).

4 See e.g. Froberg, (1969, pp. 42-3). The generalized Newton-Raphson methodology for solving dynamic programming problems used in deriving the computer solutions given in Tables 1-4 will be briefly described. Given (5), the necessary conditions for a maximum are defined by setting the partial derivatives of $\emptyset$ with respect to the four unknown parameters B_O, B_W, B_L and λ to zero. This represents a non-linear system of four equations in four unknowns. In the generalized Newton-Raphson procedure, each partial derivative equation is approximated with a Taylor expansion of order one, evaluated at some given initial point and set to zero. The result is a set of four linear equations in four unknowns. Iterative solution then proceeds with an assumption concerning the initial values of the unknown parameters until the partial derivatives evaluated at the new solution is within some specified given tolerance in absolute value. Comparison of the parameters of the computed optimal policies provides a simple check for a local optimum. The solutions given appear to provide useful estimates of the optimum.

5 See e.g., Leland (1980) and Benninga and Blume (1985).

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