
A PRACTICAL FRAMEWORK FOR PORTFOLIO CHOICE

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Traditional portfolio optimality criteria often have serious theoretical or practical limitations. A financial planning portfolio choice framework consisting of a resampled efficient portfolio set and multiperiod geometric mean analysis is a practical alternative for many situations of investment interest. While Monte Carlo financial planning is a more flexible framework, geometric mean analysis may be less error prone, theoretically justifiable, and convenient. Controversies that have limited applications of geometric mean analysis are resolvable by improved understanding of distributional properties and rational decision-making issues. The geometric mean is also useful in rationalizing a number of investment paradoxes.



Optimal portfolio choice is the central problem of equity portfolio management, financial planning, and asset allocation. Portfolio optimality in practice is typically defined relative to a Markowitz (1952, 1959) mean–variance (MV) efficient portfolio set. Markowitz or classical efficiency is computationally efficient, theoretically rigorous, and has widespread applicability. For example, Levy and Markowitz (1979) show that MV efficient portfolios are good approximations to portfolios that maximize expected utility for many utility functions and return generating processes of practical interest.¹ While there are many objections to MV efficiency, most alternatives have no less serious limitations.²

However, there are two main limitations of classical efficiency as a practical framework for optimal

portfolio choice. (1) Classical efficiency is estimation error insensitive and often exhibits poor out-of-sample performance. (2) Some additional criterion is required for portfolio choice from an efficient set. The estimation error limitations of classical efficiency and a proposed solution—the resampled efficient frontier—are detailed in Michaud (1998). The major focus of this report is to show that the distribution of the multiperiod geometric mean within a financial planning context can be the framework of choice for choosing among a properly defined efficient portfolio set for many applications of interest in investment practice.

A roadmap for the paper is as follows. A brief review of classical versus resampled MV efficiency issues for defining a practical efficient portfolio set is provided. Common optimality criteria, such as the long-term geometric mean, utility function estimation, and probability objectives, are shown to

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have significant theoretical or practical limitations. A financial planning approach, which describes the multiperiod consequences of single-period investment decisions as a framework for choosing among efficient portfolios, avoids many of the limitations of conventional and *ad hoc* optimality criteria. The pros and cons of the two main financial planning methods, Monte Carlo simulation and geometric mean analysis, are presented. The geometric mean distribution is also useful for resolving some outstanding financial paradoxes and providing valuable investment information in practice. The special case of asset allocation for defined benefit pension plans is presented. A brief summary of the results is given.

1 Classical versus resampled efficiency

Classical MV efficiency is estimation error insensitive. Jobson and Korkie (1980, 1981) show that biases in optimized portfolio weights may be very large and that the out-of-sample performance of classically optimized portfolios is generally very poor. Simple strategies like equal weighting are often remarkably superior to classical efficiency.³ In addition, classical efficiency is very unstable and ambiguous; even small changes in inputs can lead to large changes in optimized portfolio weights. Managers typically find the procedure hard to manage and often leading to unintuitive and unmarketable portfolios. The limitations of MV efficiency in practice are essentially the consequence of portfolios that are overly specific to input information. MV efficiency assumes 100% certainty in the optimization inputs, a condition never met in practice. Managers do not have perfect forecast information and find it difficult to use an optimization procedure that takes their forecasts far too literally.

Resampled efficiency uses modern statistical methods to control estimation error.⁴ Resampled optimization is essentially a forecast certainty conditional generalization of classical MV portfolio

efficiency.⁵ Statistically rigorous tests show that resampled efficient portfolios dominate the performance, on average, of associated classical efficient portfolios. In addition, managers find that resampled efficient portfolios are more investment intuitive, easier to manage, more robust relative to changes in the return generating process, and require less trading. Since investors are never 100% certain of their forecasts, there is never a good reason for an investor to use classical over resampled efficiency in practice. Unless otherwise stated, in what follows we assume that the efficient portfolio set is defined in terms of properly forecast certainty conditioned, MV resampled efficient portfolios.⁶

2 Portfolio optimality criteria

A number of portfolio optimality criteria have been proposed either based on the MV efficient set or directly. The three most common in finance literature are probably utility function estimation, short- and long-term probability objectives, and the (long-term) geometric mean. All have important theoretical or practical limitations. A brief review of the limitations of utility function and probability objective optimality criteria is provided because the issues are largely well known in the investment community. The misuses of the geometric mean are explored in more depth not only because they are less well known but also because the principles involved apply to a number of *ad hoc* optimality criteria in current investment usage.

2.1 Utility functions

Defining portfolio optimality in terms of the expectation of a utility function is the traditional finance textbook solution. Utility functions may have widely varying risk-bearing characteristics. In this approach, a utility function is chosen and its parameters estimated for a given investor or investment

situation. The set of portfolio choices may or may not be confined to portfolios on the efficient frontier. The optimal portfolio is defined as the one with maximum expected utility value.

An expected utility approach is generally not a practical investment solution for optimal portfolio choice. Investors do not know their utility function. Also, utility function estimation is very unstable. It is well known that choosing an appropriate utility function even from a restricted family of utility functions may be very difficult. In cases where a family of utility functions differs only by the value of a single parameter, even small differences of the estimated parameter may lead to very different risk-bearing characteristics (Rubinstein, 1973). Multiple-period utility functions solved with a dynamic programming or continuous-time algorithm only compound the difficulties of utility function estimation as a portfolio choice framework. As a practical matter, investors have a very difficult time explaining something as simple as why they choose one risk level over another or why risk preferences may change over time.

2.2 Short- and long-term return probabilities

The consequences of investment decisions over an investment horizon are often described in terms of the probability of meeting various short- and long-term return objectives. For example, an investor may wish to find a strategy that minimizes the probability of less than zero (geometric mean) return over a 10-year investment horizon. Other multiperiod return objectives include maximizing a positive real return or some other hurdle rate over an investment horizon. Long-term return probabilities may be approximated with the normal or lognormal distribution to the geometric mean or with Monte Carlo methods. The results are often interesting and seductively appealing. However, the tendency to define an optimal strategy based

on probability objectives, long- or short-term, has serious limitations. Markowitz (1959, p. 297) notes that return probability objectives may appear to be conservative but are often dangerous and reckless. Return probability objectives are also subject to Merton–Samuelson critiques, discussed below, and cannot be recommended.

2.3 The long-term geometric mean criterion

The geometric mean or compound return over N periods is defined as

$$G_N(\underline{r}) = \prod (1 + r_i)^{1/N} - 1, \quad (1)$$

where \underline{r} represents the vector of returns r_1, r_2, \dots, r_N in periods $1, \dots, N$, and $r_i > -1$. The usual assumptions associated with the geometric mean are that returns are measured independent of cash flows and the return generating process is independent and stationary over the investment horizon. The stationary distribution assumption is not always necessary for deriving analytical results but is convenient for many purposes.

The geometric mean is a summary statistic used in finance to describe the return over multiple equal duration discrete time intervals. Intuitively, the geometric mean statistic describes the growth rate of capital over the N -period investment horizon. It is a widely used measure of historical investment performance that is of interest to fund managers, institutional trustees, financial advisors, and sophisticated investors.

The geometric mean is usually introduced to students with the following example: Suppose an asset with a return of 100% in one period followed by -50% in the second period. The average return over the two periods is 25% but the actual return is zero. This is because a dollar has increased to two at the end of the first period and then decreased to a dollar at the end of the second. The

geometric mean formula (1) gives the correct return value, 0%. It is the measure of choice for measuring return over time. This example is pedagogically useful; it is simple, straightforward, and, within its context, correct. However, this example is easily misunderstood.

As the number of periods in the investment horizon grows large, the (almost sure) limit of the geometric mean is the point distribution:

$$G_{\infty}(r) = e^{E(\log(1+r))} - 1. \quad (2)$$

The point distribution limit (2) or long-term geometric mean is also the limit of expected geometric mean return. Formula (2) has been the source of important errors in financial literature.

Properties of the (long-term) geometric mean have fascinated many financial economists and have often been proposed as an optimality criterion.⁷ For example, the approximation for the long-term geometric mean, expressed in terms of the mean, μ , and variance, σ^2 of single-period return,

$$G_{\infty}(r) \approx \mu - \frac{\sigma^2}{2} \quad (3)$$

can be used to find the portfolio on the MV efficient frontier that maximizes long-term return.⁸ Intuitively, such a portfolio has attractive investment properties. Another optimality criterion motivated by properties of the long-term geometric mean is given in Hakansson (1971b). In this case, the criterion for portfolio optimality is

$$\text{Max } E(\log(1+r)). \quad (4)$$

As Hakansson shows, maximization of (4) leads to almost certain maximization of long-term geometric mean return while optimal MV efficient portfolios may lead to almost certain ruin.⁹ There are important theoretical and practical objections that have been raised of the Hakansson criterion (4) and its near relative (3). The theoretical objections

are discussed in the next section. From a practical point of view, the investment horizon is not infinite. For finite N , the Hakansson optimal portfolio has a variance that is often very risky. Hakansson optimal portfolios may be near, at, or beyond the maximum expected return end of the efficient frontier.¹⁰ For many investors and institutions, the Hakansson proposal is often not a practical investment objective.

2.4 Merton–Samuelson critique of the long-term geometric mean criterion

Merton and Samuelson raised serious theoretical objections to the proposals in Hakansson (1971b).¹¹ While there are a number of technical details, the basic thrust of their objections consists of the inadvisability of financial decision-making motivated by statistical properties of objective functions however intuitive or attractive. Financial decision-making must be based on expected utility maximization axioms. An objective function that is not consistent with appropriate rationality axioms leads to decisions that do not satisfy some basic rationality principle. As importantly, no one utility function is likely to be useful as a general theory of portfolio choice for all rational investors.¹²

While addressed to Hakansson (1971b), the Merton–Samuelson critiques are very general and are applicable to many *ad hoc* optimality criteria in current use in the investment community.¹³ It seems self evident that the notion of portfolio optimality and investment decision-making must necessarily rest on rationality principles similar to, if not precisely, those of classical utility.¹⁴ We assume Merton and Samuelson's views throughout our discussions.

3 Properties of the geometric mean distribution

If the number of periods is finite, the geometric mean distribution has a mean and variance and

possesses many interesting and useful properties for finance and asset management. The following simple example may provide helpful guidance. Suppose an asset with two equally probable outcomes in each investment period: 100% or -50%. What is the expected geometric mean return for investing in this asset over the investment horizon? In general it is not 0%. A correct answer requires more information.

Suppose we plan to invest in this asset for only one period. The expected return of the investment is 25% not 0%. Suppose you are considering investing in the asset for two or three investment periods. The expected geometric mean return is 12.5% over two periods and 8.26% over three periods. For any finite horizon, the investment has a variance as well as an expected return. It is only at the limit, when the number of investment periods is very large, that the expected growth rate of investing in this asset is 0%.¹⁵

An improved understanding of the properties of the geometric mean return distribution is necessary to address and resolve outstanding fallacies and to properly apply it in practice.¹⁶ Four properties of the geometric mean distribution with a focus on financial implications are given below. The reader is referred to Michaud (1981) for mathematical and statistical proofs and more technical and rigorous discussion.

3.1 *Horizon dependence*

The expected geometric mean is generally horizon dependent and monotone decreasing (or non-increasing) as the number of periods increases.¹⁷ The two-outcome example above illustrates the monotone decreasing character of the expected geometric mean and non-equality to the limit (2) when the number of periods N is finite. It is an amazingly common error, repeated in many journal papers, including finance and statistical texts, that the expected geometric mean is equal to the almost

sure limit (2) for finite N . An important corollary is that maximizing $E(\log(1+r))$ is generally not equivalent to maximizing the expected geometric mean return when N is finite. The lognormal distribution is the exception where the equality and maximization equivalence are correct.

An important consequence of this result is to highlight the often-critical limitations of the lognormal assumption for applications of geometric mean analysis. While it is easy to show that empirical asset return distributions are not normal, if only because most return distributions in finance have limited liability, it is just as easy to show that empirical asset returns are not lognormal, if only because most assets have a non-zero probability of default. Unless empirical returns are exactly lognormal, important properties of the geometric mean are ignored with a lognormal assumption. In general, lognormal distribution approximations of the geometric mean are not recommendable.¹⁸

A short digression on the related subject of continuously compounded return may be of interest. A return of 20% over a discrete time period is equal to the continuously compounded rate 18.23%. Financial researchers and practitioners often use the average of continuously compounded returns for multiperiod analyses, usually explicitly or implicitly with a lognormal distribution assumption. However, the lognormal distribution assumption is not benign; it implies horizon independence and is not consistent with most empirical returns in finance. The average of continuously compounded returns may be insufficient as a description of multiperiod return and should be used with care.

3.2 *The geometric mean normal distribution approximation*

It is well known that the geometric mean is asymptotically lognormally distributed.¹⁹ However, it is also true that it can be approximated asymptotically

by a normal distribution.²⁰ This second result turns out to have very useful applications. Asymptotic normality implies that the mean and variance of the geometric mean can be convenient for describing the geometric mean distribution in many cases of practical interest. The normal distribution can also be convenient for computing geometric mean return probabilities for MV efficient portfolios. A third important application is given in the next section.

3.3 *The expected geometric mean and median terminal wealth*

The medians of terminal wealth and of the geometric mean, G_M , are related according to the formula

$$\text{Median of terminal wealth} = (1 + G_M)^N. \quad (5)$$

Because of asymptotic normality, the expected geometric mean is asymptotically equal to the median and, consequently, the expected geometric mean is a consistent and convenient estimate of median terminal wealth via (5). Since the multiperiod terminal wealth distribution is typically highly right-skewed, the median of terminal wealth, rather than the mean, represents the more practical investment criterion for many institutional asset managers, trustees of financial institutions, and sophisticated investors.²¹ As a consequence, the expected geometric mean is a useful and convenient tool for understanding the multiperiod consequences of single-period investment decisions on the median of terminal wealth.

3.4 *The MV of geometric mean return*

A number of formulas are available for describing the N -period mean and variance of the geometric mean in terms of the single-period mean and variance of return.²² Such formulas do not typically

depend on the characteristics of a particular return distribution and range from simple and less accurate to more complex and more accurate.²³ The simplest but pedagogically most useful formulas, given in terms of the portfolio single-period mean μ and variance of return σ^2 are:

$$E(G_N(r)) = \mu - \frac{(1 - 1/N)\sigma^2}{2} \quad (6a)$$

$$V(G_N(r)) = \frac{(1 + (1 - 1/N)\sigma^2/2)\sigma^2}{N} \quad (6b)$$

Formulas (6a) and (6b) provide a useful road map for understanding the multiperiod consequences of single-period efficient investment decisions. Note that (6a) shows explicitly the horizon dependent character of expected geometric mean return.

4 **Financial planning and portfolio choice**

Financial planning methods are widely used for cash flow planning and portfolio choice in institutional consulting practice. Monte Carlo simulation and geometric mean methods are commonly associated with financial planning studies. Both methods describe the short- and long-term investment risk and return and distribution of financial consequences of investing in single-period efficient portfolios. An appropriate risk level is chosen based on visualization and assessment of the risk and return tradeoffs in financial terms for various investment horizons. Applications include defined benefit pension plan funding status and required contributions, endowment fund spending policy and fund status, investor retirement income, and college tuition trust funds. Such studies range from simply examining multiperiod return distributions and objectives to large-scale projects that include specialist consultants.²⁴ In this context, a low risk investment may often be risky relative to a higher risk alternative for meeting a specific financial goal. Financial planning methods have often been useful in identifying strategies or funding decisions that are

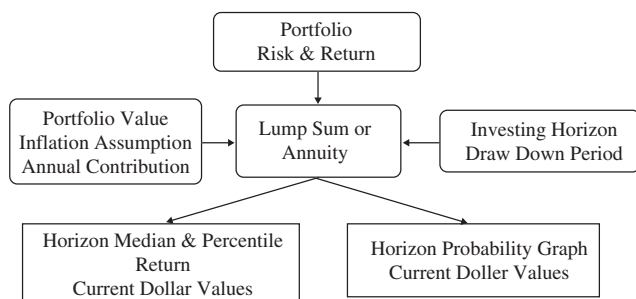


Figure 1 Financial planning framework.

likely to lead to insolvency or significant financial distress.²⁵

Figure 1 displays a standard framework for a financial planning study. The risk and return of a candidate efficient portfolio is given, capital for investment and inflation assumptions input, the length of the investment horizon and draw down period defined, and results displayed in various ways as appropriate.

4.1 Monte Carlo financial planning

Monte Carlo simulation methods are widely used for cash flow financial planning and what-if exercises. Monte Carlo methods are characterized by flexibility; virtually any cash flow computable outcome, including accounting variables and actuarial procedures, can be analyzed. Various legal and tax events are readily modeled in a Monte Carlo framework.

4.2 Geometric mean financial planning

The geometric mean distribution is also a flexible financial planning tool. Straightforward applications include planning for future college tuition, endowment and foundation asset allocation and spending rules, and 401 K pension plan retirement planning (Figure 2).²⁶ The special case of defined benefit pension plans is treated in a later

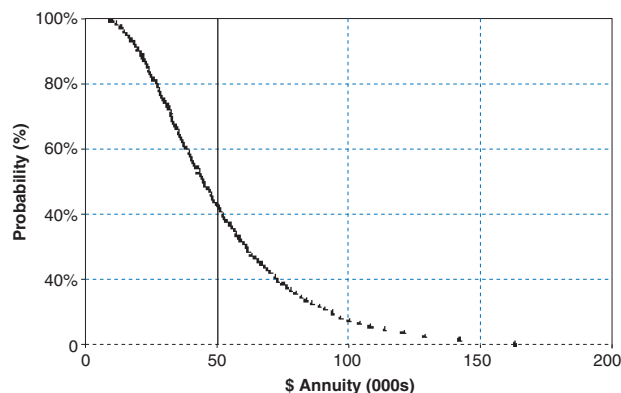


Figure 2 Twenty-year annuity distribution.

section. Variations include allowing for contributions and/or withdrawals during the investment period that may be constant or vary in value, defined either as actual cash values or as percent of fund value in each time period, in nominal or current dollars. The draw down period can be defined either in nominal or current dollars as annuities, fund values, or spending levels. Varying cash flow schedules in the contribution and draw down periods can be useful in addressing multiple objective situations.²⁷

Note that the Merton–Samuelson objections to the geometric mean as an optimality criterion are not operative in a financial planning context. As in Monte Carlo simulation, the geometric mean is simply used as a computation engine to estimate the multiperiod consequences of single-period efficient investment decisions. Properties of the geometric mean also provide the mathematical foundation of the Monte Carlo simulation financial planning process, an important issue, which we discuss further below.

4.3 Monte Carlo versus geometric mean financial planning

The advantage of Monte Carlo simulation financial planning is its extreme flexibility. Monte

Carlo simulation can include return distribution assumptions and decision rules that vary by period or are contingent on previous results or forecasts of future events. However, path dependency is prone to unrealistic or unreliable assumptions. In addition, Monte Carlo financial planning without an analytical framework is a trial and error process for finding satisfactory portfolios. Monte Carlo methods are also necessarily distribution specific, often the lognormal distribution.²⁸

Geometric mean analysis is an analytical framework that is easier to understand, computationally efficient, always convergent, statistically rigorous, and less error prone. It also provides an analytical framework for Monte Carlo studies. An analyst armed with geometric mean formulas will be able to approximate the conclusions of many Monte Carlo studies.

For many financial planning situations, geometric mean analysis is the method of choice. A knowledgeable advisor with suitable geometric mean analysis software may be able to assess an appropriate risk level for an investor from an efficient set in a regular office visit. However, in cases involving reliably forecastable path-dependent conditions, or for what-if planning exercises, supplementing geometric mean analysis with Monte Carlo methods may be required.²⁹

5 Geometric mean applications to asset management

Geometric mean properties have useful applications for asset management in situations where investment risk in each period is relatively constant over the investment horizon. This assumption is often satisfied for institutional equity strategies and many asset allocation applications and financial planning situations.

5.1 The critical point and maximum growth rates

Assume that single-period portfolio efficiency is monotone increasing in expected return as a function of portfolio risk.³⁰ Formula (6a) teaches that N -period expected geometric mean return might not be a monotone increasing function of (single-period) efficient portfolio risk.³¹ In other words, there may exist an interior “critical point” on the single-period efficient frontier that has the highest expected geometric mean return.³² This critical point can be found analytically under certain conditions or computationally using a search algorithm.³³ Institutional asset managers may often want to avoid efficient portfolios if they imply less expected geometric mean return and median wealth as well as more risk relative to others.³⁴

Figure 3 provides an example of the expected geometric mean as a function of single-period portfolio risk associated with a single-period MV efficient frontier. There are four curves. The top curve is the MV efficient frontier. The three curves below the efficient frontier display the expected geometric

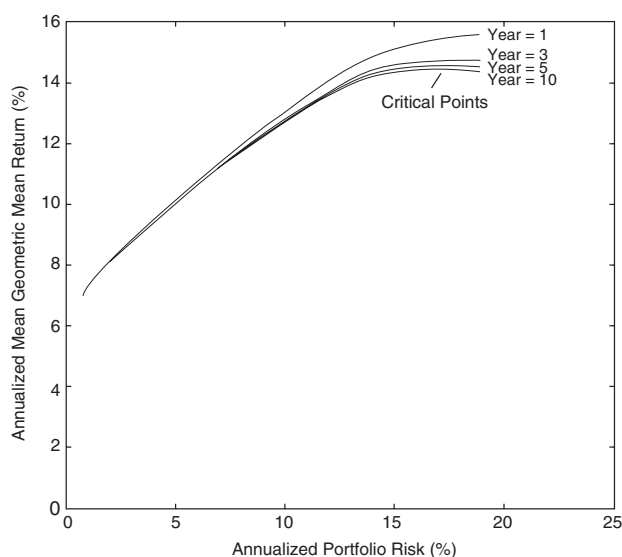


Figure 3 Efficient frontier expected geometric mean return versus portfolio risk for 1-, 3-, 5-, 10-year horizons.

mean as a function of single-period portfolio risk for three investment horizons: 3, 5, and 10 years.³⁵ Note that the expected geometric mean curves show that a critical point exists ranging roughly from 17% to 19% portfolio risk.

An interior efficient frontier critical point may not exist (Michaud, 1981). The non-existence of an interior point often means that the maximum expected geometric mean return portfolio is at the upper end point of the efficient frontier and all single-period efficient portfolios can be described as multiperiod MV geometric mean efficient.³⁶ When an interior critical point exists, it is generally horizon dependent, with limit the efficient portfolio with expected geometric mean return equal to the almost sure limit (2). The geometric mean formulas and critical point analysis can also be used to estimate an upper bound for efficient portfolio growth rates in capital markets under the assumptions.³⁷ Investors are well advised to know the multiperiod limitations of risk prior to investment, particularly when leveraged strategies are being used.

6 Resolving financial paradoxes with geometric mean analysis

A good model for investment behavior typically provides unexpected insight in totally different contexts. In this regard, the geometric mean distribution is often useful in rationalizing investment behavior and resolving paradoxes of financial management. Three examples are given below, which have interest in their own right and demonstrate the power and investment usefulness of geometric mean analysis.

6.1 CAPM and the limits of high beta portfolios

The security market line of the capital asset pricing model implies that expected return is linearly related to systematic risk as measured by β . Taken

literally, the implication is that managers should take as much β risk as they can bear. In practice, many managers do not take much more than market risk ($\beta \approx 1$) and even high-risk active portfolios seldom have a β larger than 3. Are asset managers not acting in their own and their client's best interests?

Michaud (1981) derives formulas for the critical point for β under the security market line assumption. The critical β for a market with expected annual return of 10%, risk free rate of 5%, and standard deviation of 20% for an investment horizon of 5 years is approximately 1.85. Longer horizons or larger market standard deviations lead to a smaller critical β . On the other hand, relatively recent capital market history in the US has exhibited historically low volatility and has been associated with increased popularity of leveraged hedge fund strategies. Lower market volatility, when persistent, rationalizes the use of higher leveraged strategies. In these and other situations, investment practice often mirrors the rational implications of geometric mean results.

6.2 Taxes and the benefits of diversified funds

Consider providing investment advice to an investor who owns a one stock portfolio that has performed well over a recent period. Typical financial advice is to sell the stock and buy a diversified fund. This is because the one stock portfolio has a great deal of undiversified risk. According to investment theory, diversifiable risk is not associated with long-term return and should be largely avoided.

From the investor's point of view, the advice may often not be congenial. If the stock has a larger β than the diversified fund, financial theory implies higher expected return. Also, selling the stock will certainly result in substantial capital gains taxes and loss of portfolio value. So how can the diversified fund recommendation be rationalized? This

situation is a problem encountered by financial advisors many times in their career.

The benefits of the diversified fund are not generally justifiable from single-period investment theory but often are from MV geometric mean analysis. In this context, geometric mean analysis may lead to the computation of a “crossover” point where the diversified fund is expected to outperform, and is consequently more investment attractive than, the undiversified portfolio beyond some period in the investment horizon. In many cases, the crossover point can be surprisingly short and of serious practical consideration.

Assume that the investor’s one stock portfolio has a $\beta = 2$ and a market correlation of 0.5. Assume a diversified market portfolio with expected annual return of 10% and standard deviation of 20% and a risk free rate of 5%. Assume a return generating process consistent with the security market line of CAPM and that capital gains taxes reduce capital value by 25%. Figure 4 displays the expected geometric mean return as a function of annual investment periods over a 20-period investment

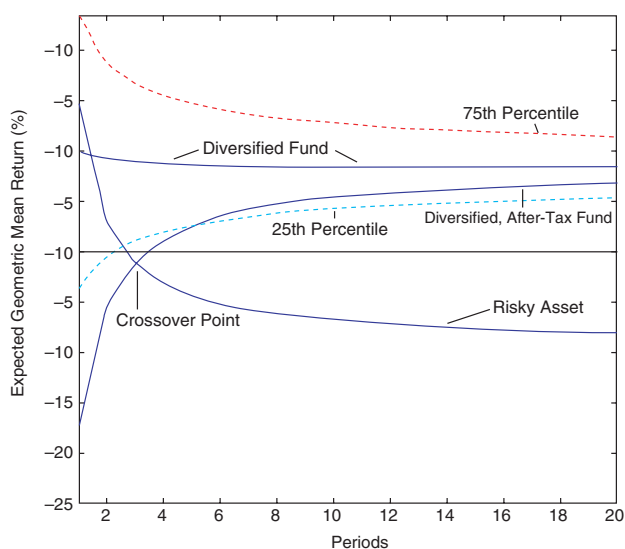


Figure 4 Crossover return analysis risky asset, diversified, diversified after-tax.

horizon for the undiversified, diversified, and diversified after-tax portfolios. In the first period, the top curve or undiversified fund has significantly higher expected return than either the middle curve (diversified fund) or bottom curve (diversified fund after-taxes). However, the exhibit shows that, over time, the expected geometric means of the diversified funds cross over and outperform the undiversified fund. This is true even when the initial loss of capital due to taxes is factored into the analysis. The diversified funds are likely to outperform the undiversified fund well within four years even considering taxes.

This example dramatically shows the power of diversification over time. It should also be noted that the example is far from extreme. Many high-performing one stock portfolios encountered in financial planning and investment consulting have β significantly in excess of 2. On the other hand, a less volatile market environment than that assumed may have significantly improved the performance of the undiversified fund.³⁸ While the results depend on the assumptions, and a crossover point need not exist, investment in diversified funds is often well rationalized by multiperiod geometric mean analysis in many cases of practical interest.³⁹

6.3 Asset allocation strategies that lead to ruin⁴⁰

Suppose an investor invests 50% of assets in risky securities in each time period. Either the return matches the investment or it is lost. Both events are equally likely. This is a fair investment game similar to an asset mix investment policy of equal allocation to risky stocks and riskless bonds with rebalancing. In this case, investment policy leads to ruin with probability one. This is because the likely outcome of every two periods results in 75% of original assets. However, the investment is always fair in the sense that the expected value of your wealth at the end of each period is always what you began with. For

two periods the expected geometric mean return is negative and declines to the almost sure long-term limit of -13.4% , which is found using (2).

This example vividly demonstrates the difference between the expected and median terminal wealth of an investment strategy. It shows that the expected geometric mean return implications of an investment decision are often of significant interest.

7 The special case of defined benefit pension plan asset allocation

Monte Carlo asset–liability simulation methods are prevalent in investment-planning practice for defined benefit pension funds. This is due to the perception that the funding of actuarially estimated liabilities and the management of actuarially estimated plan contributions is the appropriate purpose of invested assets. In this context, geometric mean analysis appears to have limited portfolio choice value. However, the traditional actuarial valuation process typically ignores the dynamic character of the economics of pension funding risk.⁴¹ These same issues make Monte Carlo asset–liability simulation studies for defined benefit pension plans often irrelevant or misleading.

7.1 *Economic nature of defined benefit pension plans*

Defining an appropriate and useful investment policy begins by understanding the true economic nature of a pension plan. A pension plan is deferred compensation. It is part of the total wage and fringe benefit package associated with employee compensation. Far from being a corporate liability or drag on firm profitability, it is a US government sponsored asset for promoting corporate competitiveness. This is because pension contributions are tax-advantaged. If the firm is to remain competitive

for human capital and total employee compensation remains the same, pension plan termination leads to greater, not less, corporate expense. Corporations should prefer employee compensation in the form of plan contributions than direct compensation.

While actuarial methods and assumptions are designed to manage the cost of the pension plan to the corporation, there are many economic forces that are at work. If total employee compensation is competitive relative to other firms, a more than normal percent of payroll plan contributions may only mean that the firm has decided to tilt total compensation towards deferred rather than current. If total compensation is high relative to competing firms, this may be part of a conscious firm policy of attracting human capital. Alternatively, there are many things the firm may want to do besides change their asset allocation in order to manage plan contributions. For example, the benefit formula, employee workforce, or level of current compensation can be reduced, all of which has direct implications for required contributions.

An appropriate asset allocation comes from an understanding of the business risks of the firm and its ability to grow and compete for human capital over time and has little, if anything, to do with actuarial valuation.⁴² A contemporary example of the dangers associated with asset allocations derived from a conventional understanding of pension liabilities is given in the next section.

7.2 *A cautionary tale for pension fund asset allocation*

As an example, the economic and market climate in the year 2001 has much to teach in terms of true economic pension liability risks and appropriate asset allocation. The year saw a dramatic decline in interest rates leading to an increase in the present value of actuarially estimated pension liabilities. At the

same time equity values fell significantly leading to a serious decline in the funding status of many US pension plans. Were large allocations to equities a terrible mistake? Should pension plans redirect their assets to fixed income instruments to reduce their funding risk in the future?

During this same period, due in part to declining equity values and associated economic conditions, many corporations downsized their workforce, froze salaries, reduced or eliminated bonuses, and shelved many internal projects. All these factors impact workforce census, expected benefits, and pension liabilities. Because the actuarial valuation process uses many non-economic long-term smoothing assumptions, liability valuation is typically little influenced by changes in expected benefits or the business risks of the firm.⁴³ An updated actuarial valuation with few smoothing assumptions, which more closely approximates financial reality, is likely to find that many US corporations had very diminished pension liabilities in this period and may be far less underfunded. Financial reality will eventually emerge from the actuarial valuation process in the form of much reduced pension liability, all other things being the same. This is because promised benefits have to be paid whatever the assumptions used to estimate them. An asset allocation based on actuarial valuation methods may often have serious negative investment consequences on plan funding when markets and economic productivity rebound and the value of non-fixed income assets become more attractive.

7.3 Economic liabilities and asset-liability asset allocation

It is beyond the scope of this report to describe the economic risk characteristics of a defined benefit pension plan or other institutional or personal liabilities and how they may be modeled.⁴⁴ Asset-liability asset allocation problems require an

understanding of changes in economic factors and capital market rates and their relationship to the economic nature of liabilities or use of invested assets.⁴⁵ Actuarial methods often have limited and even dangerous decision-making asset allocation value.

The recommended alternative is to define the resampled efficient set in a benchmark framework relative to an economic model of liability risk.⁴⁶ MV geometric mean analysis and Monte Carlo simulation may then be used to derive the multiperiod financial planning implications of efficient portfolios.

8 Conclusion

Geometric mean analysis is far more robust and applicable to a far wider range of portfolio choice applications than is widely perceived. It can rationalize much investor behavior while providing very useful information for investors and financial advisors for improving the value of invested assets. It can avoid overly risky and leveraged investments and strategies by providing investors with a realistic view of long-term capital growth rates. It is also analytically and computationally very convenient. Used properly, MV geometric mean analysis is often fundamentally important for investment consulting, financial planning, and asset management. However, the appropriate definition of the resampled efficient portfolio set remains paramount in the investment value of any financial planning procedure.

Appendix

Additional critical point issues

Formula (3) is a very standard approximation to the expected geometric mean. It has a number

of practical limitations that are shared with many other approximations in widespread use. When N is finite, the horizon dependence property illustrated in (6a) shows that the portfolio that maximizes formula (3) might not represent well the critical point portfolio. Another issue is that neither (3) nor (6a) may be sufficiently accurate approximations of $E(G_N(r))$ and the critical point when N is large. A more accurate formula from Michaud (1981, Appendix) of the long-term geometric mean return in terms of the single-period mean and variance of return, is

$$G_{\infty}(r) = (1 + \mu) \exp \left\{ - \left[\frac{\sigma^2}{2(1 + \mu)^2} \right] \right\} - 1. \quad (7)$$

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Notes

- ¹ Currently, there are serious controversies on the appropriate framework for rational decision-making under uncertainty for finance. The characteristics of investor gains and loss behavior have raised valid objections concerning the limitations of Von Neumann–Morgenstern (1953) utility axioms and alternative frameworks based on psychological principles proposed. This issue is well beyond the scope of this report. Recent research, for example Luce (2000), shows that an expanded set of utility axioms may serve as a basis for characterizing rational decision-making that addresses the gains and loss behavior objections. Luce shows that his axioms are consistent with recent psychological empirical data and competing non-axiomatic frameworks are not.
- ² Michaud (1998, Ch. 3) provides a review of the major proposed alternatives to classical efficiency and notes that classical efficiency is far more robust than is widely appreciated.

- ³ This result is a simple way to rationalize why many investors do not use classical optimization in their investment decisions.
- ⁴ Resampled efficiency, as described in Michaud (1998, Chs. 6 and 7), was co-invented by Richard Michaud and Robert Michaud and is a U.S. patented procedure, #6,003,018, December 1999, patent pending worldwide. New Frontier Advisors, LLC, has exclusive licensing rights worldwide.
- ⁵ The number of returns used to estimate simulated optimization inputs, a free parameter in the resampled efficiency process, is used to condition the optimization according to an investor's assumed level of forecast certainty. This parameter is calibrated from one to ten to facilitate the user experience. Roughly, at level one the optimized portfolios are similar to the benchmark or equal weighting; at level ten the portfolios are similar to a classical optimization. Various additional research updates of resampled efficient optimization are available at www.newfrontieradvisors.com/publications.
- ⁶ Incorporating forecast certainty as part of the definition of practical portfolio optimality is a rational, even necessary, consideration. In terms of enhanced utility axioms, Bourbaki (1948), commenting on Godel (1931), explains that rationality axioms do not characterize but follow from and codify scientific intuition. There is currently a widespread misperception in finance concerning the role of rational utility axioms and rule-based systems in scientific thought. A review of these and related issues is given in Michaud (2001). As in the case of gains and loss behavior, rule-based utility systems that do not accommodate characteristics of rational thought should be considered incomplete and reflect the need for extensions or revisions as in Luce (2000). Resampled efficiency's inclusion of forecast certainty in defining portfolio optimality is simply another case where extensions or alternative formulations of utility axioms and an enhanced notion of rational decision-making in finance are necessary.
- ⁷ An incomplete list is: Breiman (1960), Kelly (1956), Latane (1959), Markowitz (1959, Ch. 6), Hakansson (1971a,b), Thorp (1974).
- ⁸ Markowitz (1959, Ch. 6).
- ⁹ This result will be further illustrated in Section 6.3.
- ¹⁰ Hakansson (1971a) shows that the $\max E(\log(1 + r))$ portfolio may not be on the single-period classical efficient frontier.
- ¹¹ Merton and Samuelson (1974) and Samuelson and Merton (1974).

- ¹² It should be noted, as in Hakansson (1974), that the objections raised by Merton and Samuelson can be avoided by removing the statistical motivation to the argument in Hakansson (1971b). In fact, log utility is an objective function in very good expected utility axiom standing. However, without the statistical argument, log utility is simply one of many possibly interesting investment objectives.
- ¹³ A different class of *ad hoc* methods for identifying optimal portfolios has to do with questionnaires that investors are asked to answer that purport to measure risk preferences and result in a recommended “model” portfolio from a predefined set. Such methods typically have no theoretical justification and may provide little, if any, reliable or useful information for investors.
- ¹⁴ Von Neumann and Morgenstern (1953).
- ¹⁵ In this case 0% is also the almost sure limit of the geometric mean.
- ¹⁶ A surprisingly widespread simple asset allocation error is to use geometric instead of arithmetic mean inputs in a classical optimization to moderate the effect of large return and risk assets and make the solutions more acceptable to investors. Stein methods, discussed in Michaud (1998, Ch. 8) are often the appropriate methods for shrinking outlier data for the purpose of improving forecastability.
- ¹⁷ A statistic may or may not be dependent on the number of observations in a sample. Examples include sample size independence of the sample mean and sample size dependence of the sample variance.
- ¹⁸ Unless otherwise noted, our results in the following are non-parametric and do not depend on the lognormal return distribution assumption.
- ¹⁹ Applying the log function to each side of the equality (1) and invoking the central limit theorem implies that the N -period geometric mean distribution is asymptotically lognormal.
- ²⁰ The fact that a distribution can asymptotically be well approximated by two different distributions is not unique in probability theory. The binomial distribution can be approximated asymptotically by both the normal and Poisson distribution under certain conditions. Intuitively, a lognormal characterization of the asymptotic geometric mean return distribution may seem more natural because of the skewness normally associated with multiperiod returns. However, the N th root function reduces much of the skewness effect when N is reasonably large.
- ²¹ The relationship between N -period geometric mean return and terminal wealth is given by: $W_N(r) = (1 + G_N(r))^N = \prod (1 + r_i)$. Applying the log function to each side of the equality and invoking the central limit theorem leads to the conclusion that N -period terminal wealth is asymptotically lognormal.
- ²² For example, Young and Trent (1969).
- ²³ Michaud (1981) provides caveats on the applicability and approximation accuracy of these and other formulas.
- ²⁴ One early comprehensive Monte Carlo study of pension fund investment policy that included an examination of the volatility of pension liabilities under workforce census changes, corporate policy, and market rate assumptions is given in Michaud (1976).
- ²⁵ The author first encountered this effect in 1974 when conducting a Monte Carlo simulation study of proposed spending and risk policies for the Harvard College Endowment Fund. Under some proposals that were subsequently rejected, the simulations showed that the fund may have run out of money within roughly twelve years. Multiperiod insolvency cases were also encountered in Monte Carlo studies for individuals that proposed to spend capital at unsustainable rates.
- ²⁶ For example: A prospective retiree has \$500,000 to invest for retirement. There are ten years until retirement. The fund has an expected return of 10% and a 20% standard deviation. The goal is to purchase a retirement annuity that will provide \$50,000 annual income in constant dollar terms. A life expectancy of 20 years in retirement and a 3% inflation rate is assumed. What is the likelihood of the \$50 K annuity and median annuity value at retirement? Using simple annuity formulas, a geometric mean analysis shows that there is a 43% chance of reaching the \$50,000 annuity objective for a 20-year period in retirement with a median value of \$45,000. The 20-year distribution of annuity values and probabilities are displayed in Figure 2. A less risky strategy of 7% portfolio return and 10% standard deviation leads to a 17% probability of meeting the \$50,000 annuity objective with a median annuity value of \$38,000.
- ²⁷ The assumption that allows geometric mean analysis to address these and other long-term investment planning issues and multiple objectives is that the consequence of cash flows leaves the underlying return generating process unchanged. Adjustment for the impact of intermediate cash flows is implemented using multiple geometric mean investment horizon assumptions.
- ²⁸ Limitations of the lognormal assumption were described in Section 3.1.
- ²⁹ Many tax and legal situations are extremely complicated. Often the only available solutions for cash flow planning are heuristics that have evolved from experience and

insight. In such cases, Monte Carlo methods may be the only recourse. Also the impact of trading decisions and costs over time may only be resolvable with Monte Carlo methods. In these and other cases, geometric mean analysis followed by detailed Monte Carlo simulation, assuming economic feasibility, is the recommended procedure.

- ³⁰ Unlike classical efficiency, the resampled efficient frontier may curve downward from some point and may not be monotone increasing in expected return as a function of portfolio risk. The investment implications include limitations of high-risk assets not well represented by classical efficiency.
- ³¹ Markowitz (1959, Ch. 6) noted this possibility from his simulations relative to the geometric mean limit formula (3).
- ³² While efficient frontiers in practice often satisfy a budget constraint and non-negative portfolio weights, neither resampled efficiency nor geometric mean critical point analysis is limited to such frontiers. In particular, a critical point can be computed for unbounded leverage efficient frontiers as in Hakansson (1971a) and can be very revealing.
- ³³ Michaud (1981) provides analytical solutions for the critical point in terms of portfolio β .
- ³⁴ It should be emphasized that the critical point is a total, not residual, risk–return geometric mean concept.
- ³⁵ The efficient frontier is based on annualized historical monthly return Ibbotson Associates (Chicago, IL) index data for six asset classes – T-Bills, intermediate government bonds, long-term corporate bonds, large capitalization U.S. equity, small capitalization U.S. equity, international equity – from January 1981 to December 1993. See Appendix A for additional critical point issues.
- ³⁶ The exceptional case is given in Hakansson (1971a) where the critical point is at the origin.
- ³⁷ This result is given in Michaud (1981).
- ³⁸ An all or nothing trading strategy is not the only way to implement a multiperiod diversification program. Roughly, the same principles apply to diversifying a fixed amount of capital over multiple periods in order to manage trading and other costs.
- ³⁹ The tax effect could have been dealt with in a number of ways. It is unlikely that many investors would convert 100% of a one stock portfolio into a diversified fund in the first period. However, the tax effect is something of an illusion. Unless taxes can be avoided in some way altogether, the one stock portfolio is likely to be subject to tax at some point in the investment horizon and the comparison may be even more favorable for diversified funds than illustrated.

⁴⁰ From Block (1969).

⁴¹ For example, Michaud (1979) notes the irrelevance of the widely used actuarial interest rate in defined benefit plans as an investment objective for asset allocation.

⁴² See Michaud (1998, Ch. 10) for further discussion of these issues.

⁴³ The standard rationale for smoothing assumptions is that required contributions should not be affected greatly by relatively ephemeral volatility in capital markets. However, this argument has the critical flaw that the firm's business risks do not exist in isolation to capital market volatility or changes in the domestic and global economy.

⁴⁴ An important issue, ignored here, is the impact of the 50% nondeductible reversion tax now assessed on excess assets from a terminated US defined benefit pension plan. These taxes alter the economics of pension plan liability risk. See Ippolito (2002) for further discussion.

⁴⁵ These and related issues are discussed further in Michaud (1998, Ch. 10).

⁴⁶ See Michaud (1998, Ch. 10).

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